Improved high-temperature expansion and critical equation of state of three-dimensional Ising-like systems

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High-temperature series are computed for a generalized three-dimensional Ising model with arbitrary potential. Three specific ''improved'' potentials (suppressing leading scaling corrections) are selected by Monte Carlo computation. Critical exponents are extracted from high-temperature series specialized to improved potentials, achieving high accuracy; our best estimates are $\gamma=1.2371(4)$, $\nu=0.630\,02(23)$, $\alpha=0.1099(7)$, $\eta=0.0364(4)$, $\beta=0.326\,48(18)$. By the same technique, the coefficients of the small-field expansion for the effective potential (Helmholtz free energy) are computed. These results are applied to the construction of parametric representations of the critical equation of state. A systematic approximation scheme, based on a global stationarity condition, is introduced (the lowest-order approximation reproduces the linear parametric model). This scheme is used for an accurate determination of universal ratios of amplitudes. A comparison with other theoretical and experimental determinations of universal quantities is presented.
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I. INTRODUCTION

According to the universality hypothesis, critical phenomena can be described by quantities that do not depend on the microscopic details of a system, but only on global properties such as the dimensionality and the symmetry of the order parameter. Many three-dimensional systems characterized by short-range interactions and a scalar order parameter (such as density or uniaxial magnetization) belong to the Ising universality class. This implies that the critical exponents, as well as other universal quantities, are the same for all these models. Their precise determination is therefore important in order to test the universality hypothesis.

The high-temperature (HT) expansion is one of the most effective approaches to the study of critical phenomena. Much work (even recently) has been devoted to the computation of HT series, especially for N-vector models and in particular the Ising model. An important issue in the analysis of the HT series is related to the presence of nonanalytic corrections to the leading power-law behavior. For instance, according to the renormalization-group theory (see, e.g., Ref. [1]), the magnetic susceptibility should behave as

$$\chi=Ct^{-\gamma}(1+a_{0,1}t+a_{0,2}t^2+\cdots+a_{1,1}t^{\Delta}+a_{1,2}t^{2\Delta}+\cdots+a_{2,1}t^{\Delta_2}+\cdots), \tag{1.1}$$

where $t\equiv(T-T_c)/T_c$ is the reduced temperature. The leading critical exponent γ, and the correction exponents $\Delta,\Delta_2,\ldots$, are universal, while the amplitudes C and $a_{i,j}$ are not universal and should depend smoothly on any subsidiary parameter that may change T_c, but does not affect the nature of the transition. Nonanalytic correction terms to the leading power-law behavior, represented by noninteger powers of t, are related to the presence of irrelevant operators. For three-dimensional Ising-like systems, the existence of leading corrections with exponent $\Delta\simeq0.5$ is well established. In order to obtain precise estimates of the critical parameters, the approximants of the HT series should properly allow for the confluent nonanalytic corrections [2–8]. The so-called integral approximants [9] can, in principle, allow for them (see, e.g., Ref. [10] for a review). However, they require long series to detect nonleading effects, and in practice they need to be biased to work well. Analyses meant to effectively allow for confluent corrections are generally based on biased approximants where the value of β_c and the first nonanalytic exponent Δ is given (see, e.g., Refs. [11–15]). It is indeed expected that the leading nonanalytic correction is the dominant source of systematic error.

An alternative approach to this problem is the construction of a HT expansion where the dominant confluent correction is suppressed. If the leading nonanalytic terms are no longer present in the expansion, the analysis technique based on integral approximants should become much more effective, since the main source of systematic error has been eliminated. In order to obtain an improved high-temperature (IHT) expansion, we may consider improved Hamiltonians characterized by a vanishing coupling with the irrelevant operator responsible for the leading scaling corrections. This idea has been pursued by Chen, Fisher, and Nickel [5], who studied classes of two-parameter models (such as the bcc scalar double-Gaussian and Klauder models). Such models interpolate between the spin-1/2 Ising model and the Gaussian model, and they are all expected to belong to the Ising universality class. The authors of Ref. [5] showed that improved models with suppressed leading corrections to scaling can be obtained by tuning the parameters (see also Refs. [8,16]). This approach has been recently considered in the

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context of Monte Carlo simulations [17–20], using lattice ϕ^4 models. It is also worth mentioning the recent work [21] of Belohorec and Nickel in the context of dilute polymers, where a substantial improvement in the determination of the critical exponents ν and ω was achieved by simulating the two-parameter Domb-Joyce model.

We consider the class of scalar models defined on the simple cubic lattice by the Hamiltonian

$$\mathcal{H}=-\beta\sum_{\langle i,j\rangle}\phi_i\phi_j+\sum_i V(\phi_i^2), \tag{1.2}$$

where $\beta\equiv 1/T$, $\langle i,j\rangle$ indicates nearest-neighbor sites, ϕ_i are real variables, and $V(\phi^2)$ is a generic potential (satisfying appropriate stability constraints). The critical limit of these models is expected to belong to the Ising universality class (apart from special cases corresponding to multicritical points). Using the linked cluster expansion technique, we calculated the high-temperature expansion to 20th order for an arbitrary potential, generalizing the existing expansions for the standard Ising model (see, e.g., Ref. [13] for a review of the existing HT calculations). In this work we will essentially consider and present results for a potential of the form

$$V(\phi^2)=\phi^2+\lambda_4(\phi^2-1)^2+\lambda_6(\phi^2-1)^3; \tag{1.3}$$

such a potential will be assumed in the following, unless otherwise stated. Within this family of potentials, improved Hamiltonians can be obtained by looking for values of the parameters λ_4 and λ_6 for which leading scaling corrections are suppressed. In particular we may keep λ_6 fixed and look for the corresponding value λ_4^* of λ_4 that gives an improved Hamiltonian. Notice that for generic choices of the Hamiltonian λ_4^* may not exist. This is the case of the $O(N)$ ϕ^4 theory with nearest-neighbor couplings on a cubic lattice in the large-N limit, where it is impossible to find a positive value of λ_4 achieving the suppression of the dominant scaling corrections. Using the estimates of the leading scaling correction amplitudes reported in Ref. [15], one can argue that the same is true for finite $N>3$. As shown numerically by Monte Carlo simulations [17–20], λ_4^* exists in the case $N=1$, which is the single-component ϕ^4 model (i.e., the model presented above with $\lambda_6=0$). By using finite-size techniques, Hasenbusch obtained a precise estimate of λ_4^*: $\lambda_4^*=1.10(2)$ [20]. In our work we will also consider the spin-1 (or Blume-Capel) Hamiltonian

$$\mathcal{H}=-\beta\sum_{\langle i,j\rangle}s_is_j+D\sum_i s_i^2, \tag{1.4}$$

where the variables s_i take the values $0,\pm 1$. In this case the value of D for which the leading scaling corrections are suppressed is $D^*=0.641(8)$ [22].

From the point of view of the HT expansion technique, the main problem is the determination of the improved Hamiltonian. Once the improved Hamiltonian is available, the analysis of its HT series leads, as we shall see, to much cleaner and therefore reliable results. A precise estimate of the parameters associated with an improved Hamiltonian is crucial in order to obtain a substantial improvement of the IHT results. As shown in Refs. [17,20], Monte Carlo simulations using finite-size scaling techniques seem to provide the most efficient tool for this purpose. For comparison, in the case of the pure ϕ^4 theory ($\lambda_6=0$), our best estimate of λ_4^* from the HT expansion is consistent with the above-mentioned Monte Carlo result $\lambda_4^*=1.10(2)$, but it is affected by an uncertainty of about 10% (see Sec. III). So we decided to follow the strategy of determining the improved Hamiltonian by Monte Carlo simulations employing finite-size scaling techniques. For the ϕ^4 model, this work has been satisfactorily done by Hasenbusch [20]. For the ϕ^6 model with $\lambda_6=1$, we performed Monte Carlo simulations in order to calculate λ_4^*, obtaining $\lambda_4^*=1.90(4)$.

The comparison of the results obtained from the three improved Hamiltonians considered strongly supports our working hypothesis of the reduction of systematic errors in the IHT estimates, and provides an estimate of the residual errors due to the subleading confluent corrections to scaling.

The analysis of our 20th-order IHT series allows us to obtain very precise estimates of the critical exponents γ, ν and η. Our estimates substantially improve previous determinations by HT and other methods.

We extended our study to the small-field expansion of the effective potential, which is the Helmholtz free energy of the model. This expansion can be parametrized in terms of the zero-momentum n-point couplings g_n in the symmetric phase. The analysis of the IHT series provides new results for the couplings g_n, and leads to interesting comparisons with the estimates from other approaches based on field theory and lattice techniques. Moreover, we improved the knowledge of the universal critical low-momentum behavior of the two-point function of the order parameter, which is relevant for critical scattering phenomena.

By exploiting the known analytic properties of the critical equation of state, one may reconstruct the full critical equation of state from the small-field expansion of the effective potential, which is related to the behavior of the equation of state for small magnetization in the symmetric phase. This can be achieved by using parametric representations implementing in a rather simple way the known analytic properties of the equation of state. Effective parametric representations can be obtained by parametrizing the magnetization M and the reduced temperature t in terms of two variables R and θ, setting $M\propto R^\beta\theta$, $t=R(1-\theta^2)$, and $H\propto R^{\beta\delta}h(\theta)$. In this framework, following Guida and Zinn-Justin [23], one may develop an approximation scheme based on truncations of the Taylor expansion of the function $h(\theta)$ around $\theta=0$. Knowing a given number of terms in the small-field expansion of the effective potential, one can derive the same number of terms in the small-θ expansion of $h(\theta)$, with a dependence on an arbitrary normalization parameter ρ. One can try to fix ρ so that this small-θ expansion has the fastest possible convergence. We propose a prescription based on the global stationarity of the truncated equation of state with respect to the arbitrary parameter ρ. This extends the stationarity condition of the linear model (i.e., the lowest-order nontrivial approximation) discussed in Refs. [24–27]. Using the IHT results for γ, ν and the first few coefficients of the small-field expansion of the effective potential, we constructed approximate representations of the full critical equation of state. From them we obtained accurate estimates of many ratios of universal amplitudes. Varying the truncation order of $h(\theta)$, we observed a fast convergence, supporting our arguments.

For our readers' convenience, we collected in Table XIII a summary of all the results obtained in this paper. There they can find new estimates of most of the universal quantities (exponents and ratios of amplitudes) introduced in the literature to describe critical phenomena in three-dimensional (3D) Ising-like systems. The only important quantity for which we have not been able to give a good estimate is the exponent ω, which is related to the leading scaling corrections. We mention that a precise estimate of ω has been reported recently in Ref. [20]: $\omega=0.845(10)$. It has been obtained by a Monte Carlo study using a finite-size scaling method.

The paper is organized as follows. In Sec. II we discuss the main features of improved Hamiltonians from the point of view of the renormalization group. In Sec. III we describe our Monte Carlo simulations and present the estimates of λ_4^* for the potential (1.3) with $\lambda_6=1$. Section IV is dedicated to the determination of the critical exponents from the IHT expansion. We present estimates of all relevant critical exponents (except for ω), and compare our results with other theoretical approaches and experiments. In Sec. V we study the small-field expansion of the effective potential. We present estimates of the first few coefficients of the expansion. We discuss the relevance of the determination of the zero-momentum four-point renormalized coupling for field-theoretical approaches (Sec. V B). Section VI presents a study of the low-momentum behavior of the two-point function in the critical region. Estimates of the first few coefficients of its universal low-momentum expansion are given. In Sec. VII we study the critical equation of state, which gives a description of the whole critical region, including the low-temperature phase. Using the estimates of the critical exponents and of the first few coefficients of the small-field expansion of the effective potential, the critical equation of state is reconstructed employing approximate parametric representations (Sec. VII A). In Sec. VII B we present our method, based on the global stationarity of the approximate equation of state. Relevance to the ϵ expansion is discussed in Sec. VII C. In Sec. VII D we apply the results of Sec. VII B to the computation of universal ratios of amplitudes, using as inputs the results of the IHT expansion. The results are then compared with other theoretical estimates and with experimental determinations. For the sake of comparison, we also present results for the two-dimensional Ising model.

Many details of our calculations are reported in the Appendices. Appendix A contains a detailed description of our HT calculations, i.e., the list of the quantities we have considered and the description of the method we used to generate and analyze the HT series. We report many details and intermediate results so that the reader can judge the quality of the results we will present. In Appendix B we present the notations for the critical amplitudes, and report the expressions of the universal ratios of amplitudes in terms of the parametric representation of the critical equation of state. In Appendix C we discuss in more detail the approximation scheme for the parametric representation of the equation of state based on stationarity.

II. IMPROVED HAMILTONIANS

As discussed in the Introduction, we will work with ''improved'' Hamiltonians, i.e., with models in which the leading correction to scaling has a vanishing (in practice very small) amplitude.

To clarify the basic idea, let us consider a model with two relevant operators (the thermal and the magnetic ones) and one irrelevant operator. If τ, κ, and μ are the associated nonlinear scaling fields, the singular part of the free energy $F_{\rm sing}$ has the scaling form [28]

$$F_{\rm sing}(\tau,\kappa,\mu)=|\tau|^{d\nu}f_{\pm}(\kappa|\tau|^{-(d+2-\eta)\nu/2},\mu|\tau|^{\Delta}), \quad (2.1)$$

where the function $f_{\pm}$ depends on the phase of the model. Since the operator associated with μ is irrelevant, Δ is positive and $\mu|\tau|^{\Delta}\to 0$ at the critical point. Therefore, we can expand the free energy, obtaining

$$F_{\rm sing}(\tau,\kappa,\mu)=|\tau|^{d\nu}\sum_{n=0}^{\infty} f_{n,\pm}(\kappa|\tau|^{-(d+2-\eta)\nu/2})\mu^n|\tau|^{n\Delta}. \quad (2.2)$$

The presence of the irrelevant operator induces nonanalytic corrections proportional to $|\tau|^{n\Delta}$. Now, let us suppose that the Hamiltonian of our model depends on three parameters r, h, and λ, where r is associated with the temperature, h is the magnetic field, and λ is an irrelevant parameter. For each value of λ and for $h=0$, the theory has a critical point for $r=r_c(\lambda)$. The nonlinear scaling fields τ, κ, and μ are analytic functions of the parameters appearing in the Hamiltonian, and therefore we can write

$$\tau=t+t^2g_{1\tau}(\lambda)+h^2g_{2\tau}(\lambda)+O(t^3,th^2,h^4), \quad (2.3)$$

$$\kappa=h[1+tg_{1\kappa}(\lambda)+h^2g_{2\kappa}(\lambda)+O(t^2,th^2,h^4)], \quad (2.4)$$

$$\mu=g_{1\mu}(\lambda)+tg_{2\mu}(\lambda)+h^2g_{3\mu}(\lambda)+O(t^2,th^2,h^4), \quad (2.5)$$

where $t\equiv r-r_c(\lambda)$. Substituting these expressions into Eq. (2.2), we see that, if $g_{1\mu}(\lambda)\neq 0$, the free energy has corrections of order $t^{n\Delta}$. For the susceptibility in zero magnetic field we obtain the explicit formula [29]

$$\chi=t^{-\gamma}\sum_{m,n=0}^{\infty}\chi_{1,mn}(\lambda)t^{m\Delta+n}+t^{1-\alpha}\sum_{m,n=0}^{\infty}\chi_{2,mn}(\lambda)t^{m\Delta+n} + \sum_{n=0}^{\infty}\chi_{3,n}(\lambda)t^n, \quad (2.6)$$

where the contribution proportional to $t^{1-\alpha}$ stems from the terms of order h^2 appearing in the expansion of τ and μ, and the last term is the contribution of the regular part of the free energy. Notice that it is often assumed that the regular part of the free energy does not depend on h. If this were the case, we would have $\chi_{3,n}(\lambda)=0$. However, for the two-dimensional Ising model, one can prove rigorously that $\chi_{3,0}\neq 0$ [30,31], showing the incorrectness of this conjecture. For a discussion, see Ref. [32].

In many interesting instances, it is possible to cancel the leading correction due to the irrelevant operator by choosing

$\lambda=\lambda^*$ such that $g_{1\mu}(\lambda^*)=0$. In this case $\mu\tau^\Delta\sim t^{1+\Delta}$, so that no term of the form $t^{m\Delta+n}$, with $n<m$, will be present. In particular, the leading term proportional to t^Δ will not appear in the expansion.

In general, other irrelevant operators will be present in the theory, and therefore we expect corrections proportional to t^ρ with $\rho=n_1+n_2\Delta+\Sigma_i m_i\Delta_i$, where Δ_i are the exponents associated with the additional irrelevant operators. For $\lambda=\lambda^*$ the expansion will contain only terms with $n_1\geqslant n_2$.

It is important to note that by working with $\lambda=\lambda^*$, we use a Hamiltonian such that the nonlinear scaling field μ vanishes at the critical point. This property is independent of the observable we are considering. Therefore, all quantities will be improved, in the sense that the leading correction to scaling, proportional to t^Δ, will vanish. We will call the Hamiltonians with $\lambda=\lambda^*$ ''improved Hamiltonians.''

III. DETERMINATION OF THE IMPROVED PARAMETERS

The Hamiltonian defined by Eqs. (1.2) and (1.3) with $\lambda_6=0$ was considered in Ref. [20], where it was shown that the leading correction to scaling cancels for $\lambda_4^*=1.10(2)$. Here we will also consider the case $\lambda_6=1$, and determine the corresponding λ_4^* using a method similar to the one discussed in Ref. [17].

The idea is the following. Consider a renormalization-group invariant observable $\mathcal{O}$ on a finite lattice L and let $\mathcal{O}^*$ be its value at the critical point, i.e.,

$$\mathcal{O}^*=\lim_{L\to\infty}\lim_{\beta\to\beta_c(\lambda_4)}\mathcal{O}(\beta,\lambda_4,L). \tag{3.1}$$

The quantity $\mathcal{O}^*$ is a universal number and therefore it will be independent of λ_4. The standard scaling arguments predict

$$\mathcal{O}(\beta_c(\lambda_4),\lambda_4,L)\approx\mathcal{O}^*+a_1(\lambda_4)L^{-\omega}+a_2(\lambda_4)L^{-2\omega}+\cdots +b_1(\lambda_4)L^{-\omega_2}\cdots, \tag{3.2}$$

where $\omega=\Delta/\nu$, $\omega_2=\Delta_2/\nu$, Δ_2 being the next-to-leading correction-to-scaling exponent. Since for $\lambda_4=\lambda_4^*$, $a_1(\lambda_4^*)=a_2(\lambda_4^*)=\cdots=0$, for $\lambda_4\approx\lambda_4^*$ we can rewrite the previous equation as

$$\mathcal{O}(\beta_c(\lambda_4),\lambda_4,L)\approx\mathcal{O}^*+(\lambda_4-\lambda_4^*)(a_{11}L^{-\omega}+a_{21}L^{-2\omega}+\cdots)+b_1(\lambda_4^*)L^{-\omega_2}\cdots. \tag{3.3}$$

Now, suppose we know the exact value $\mathcal{O}^*$, and let us define $\lambda_4^{\rm eff}(L)$ as the solution of the equation

$$\mathcal{O}(\beta_c(\lambda_4^{\rm eff}(L)),\lambda_4^{\rm eff}(L),L)=\mathcal{O}^*. \tag{3.4}$$

From Eq. (3.3) we obtain immediately

$$\lambda_4^{\rm eff}(L)=\lambda_4^*-\frac{b_1(\lambda_4^*)}{a_{11}}L^{\omega-\omega_2}+\cdots. \tag{3.5}$$

Since $\omega_2>\omega$, $\lambda_4^{\rm eff}(L)$ converges to λ_4^* as $L\to\infty$. For the 3D Ising universality class, $\omega_2\simeq 2\omega$ [33,34] and $\omega\simeq 0.85$ [20].

TABLE I. For several values of lattice size L and for $\lambda_6=0$, we report the values of the parameters used in the Monte Carlo simulation $\lambda_{4,\rm run}$, $\beta_{\rm run}$, the number of Monte Carlo iterations $N_{\rm iter}$, each iteration consisting of a standard Swendsen-Wang update and of a Metropolis sweep, and the estimate of the Binder parameter Q at $\lambda=1.10$, $\beta=0.3750973$. The reported error on Q is the sum of three terms: the statistical error, and the errors due to the uncertainty of λ^* and $\beta_c(\lambda)$.

L	$\lambda_{4,\rm run}$	$\beta_{\rm run}$	$N_{\rm iter}$	Q
6	1.100	0.375	91×10^6	0.62370(15+37+2)
7	1.100	0.375	78×10^6	0.62386(17+33+2)
9	1.080	0.376	178×10^6	0.62389(12+26+3)
12	1.105	0.375	305×10^6	0.62387(10+25+5)

In order to apply this method in practice we need two ingredients: a precise determination of $\beta_c(\lambda_4)$ and an estimate of $\mathcal{O}^*$.

Very precise estimates of $\beta_c(\lambda_4)$ can be obtained from the analysis of the HT series of the susceptibility χ, which we have calculated to $O(\beta^{20})$. For $\lambda_6=0$ and $1.0\leqslant\lambda_4\leqslant1.2$ the values of $\beta_c(\lambda_4)$ can be interpolated by the polynomial

$$\beta_c(\lambda_4)=0.40562043+0.00819000\lambda_4-0.04626355\lambda_4^2+0.01235674\lambda_4^3\pm0.0000014. \tag{3.6}$$

In particular, for $\lambda_4=1.10$, we have $\beta_c(1.10)=0.3750973(14)$, to be compared with $\beta_c(1.10)=0.3750966(4)$ of Ref. [20]. For $\lambda_6=1$ and $1.8\leqslant\lambda_4\leqslant2.0$ — as we shall see, this is the relevant interval — we have

$$\beta_c(\lambda_4)=0.68612192-0.18274273\lambda_4+0.02634688\lambda_4^2-0.00102710\lambda_4^3\pm0.0000018. \tag{3.7}$$

The second quantity we need is an observable $\mathcal{O}$ such that $\mathcal{O}^*$ can be computed with high precision. We have chosen the Binder parameter

$$Q=\frac{\langle m^4\rangle}{\langle m^2\rangle^2}, \tag{3.8}$$

where m is the magnetization. A precise estimate of Q was obtained in Ref. [17] by means of a large-scale simulation of the spin-1 model. They report

$$Q^*=0.62393(13+35+5), \tag{3.9}$$

where the error is given as the sum of three contributions: the first is the statistical error; the second and the third account for corrections to scaling. We have tried to improve this estimate by performing a high-precision Monte Carlo simulation of the Hamiltonian (1.2) for $\lambda_6=0$ and by computing Q for $\lambda_4=1.10$, which is the best estimate of λ_4^*. We used the Brower-Tamayo algorithm [35], each iteration consisting of a Swendsen-Wang update of the sign of ϕ and of a Metropolis sweep. Since the Hamiltonian (1.2) is improved (i.e., the leading correction to scaling vanishes), we expect to be able to obtain a reliable estimate from simulations on small lattices for which it is possible to accumulate a large statistics. The results are reported in Table I. (The simulations

TABLE II. For several values of lattice size L and for $\lambda_6=1$, we report the values of the parameters used in the Monte Carlo simulation $\lambda_{4,\mathrm{run}}, \beta_{\mathrm{run}}$, the number of Monte Carlo iterations N_{iter}, each iteration consisting of a standard Swendsen-Wang update and of a Metropolis sweep, and the estimate of $\lambda^{\mathrm{eff}}(L)$, the solution of Eq. (3.4). The error is reported as the sum of three terms: the statistical error, and the errors due to the uncertainty of Q^* and $\beta_c(\lambda)$.

L	$\lambda_{4,\mathrm{run}}$	β_{run}	N_{iter}	$\lambda_4^{\mathrm{eff}}(L)$
6	1.900	0.427	100×10^6	1.915(9+19+1)
7	1.900	0.427	252×10^6	1.894(6+21+2)
9	1.920	0.425	214×10^6	1.897(9+25+3)
12	1.920	0.425	294×10^6	1.904(9+32+6)

were performed before the appearance of Ref. [20] and the generation of the HT series, when only a very approximate expression for λ_4^* existed. Therefore, the runs were not made at the correct values of λ_4 and β_c. The values reported in Table I have been obtained from the Monte Carlo data by means of a standard reweighting technique.)

There are three different sources of error: we report the statistical error, the variation of the estimate of the Binder parameter when λ_4 varies within the interval $1.08-1.12$ (due to corrections to scaling of order $L^{-\omega}$, which are not completely suppressed, since the value used for λ_4 is not exactly equal to λ_4^*), and the variation of Q when β_c varies within one error bar. The values of L that we use are relatively small ($L\leqslant12$) and one could fear that next-to-leading corrections still give a non-negligible systematic deviation. Our data do not show any evidence of such an effect, and the estimates for different values of L are consistent. Using the estimate obtained for $L=12$, we get the final result

$$Q^*=0.62388(32) \tag{3.10}$$

(the uncertainty is obtained assuming independence of systematic and statistical errors), which is in agreement with the estimate (3.9) with a slightly smaller error bar.

We have next determined λ_4^* for the model with Hamiltonian (1.2) and $\lambda_6=1$ using the method presented above. Estimates of $\lambda_4^{\mathrm{eff}}(L)$ are reported in Table II, from which we conclude

$$\lambda_4^*=1.90(4). \tag{3.11}$$

Note that the last three points show a small upward trend which, although consistent with a statistical effect, could be a systematic increase due to the corrections of order $L^{-\omega_2+\omega}$ or could be due to the fact that Q^* is only approximately known. To exclude the latter case, we have computed $\lambda_{4,\pm}^{\mathrm{eff}}(L)$, the solutions of Eq. (3.4) with the rhs replaced by $Q^*\pm\sigma_Q$, σ_Q being the error on Q^*. If the increase is associated with the uncertainty of Q^*, we should observe that $\lambda_{4,\pm}^{\mathrm{eff}}(L)$ have opposite trends, one increasing, the other decreasing. In the present case $\lambda_{4,\pm}^{\mathrm{eff}}(L)$ are both increasing, thereby excluding the possibility that this effect is due to the uncertainty of Q^*. With the present statistical errors we cannot distinguish between the first two possibilities: we have considered as our final estimate the average of the results obtained for $L=7,9,12$, but we cannot exclude the possibility that the correct λ_4^* is slightly larger than our estimate. However, the quoted error should be large enough to include this systematic increase.

As explained in the Introduction, the analysis of HT series for the purpose of determining universal quantities is sensitive to nonanalytic scaling corrections. As we will discuss below, one can use this fact to obtain a rough estimate of the optimal value of λ.

Consider, for example, the zero-momentum four-point coupling constant g_4 defined by

$$g_4=-\frac{\chi_4}{\chi^2\xi^3}, \tag{3.12}$$

where χ, ξ^2, and χ_4 are, respectively, the magnetic susceptibility, the second-moment correlation length, and the zero-momentum four-point connected correlation function (definitions can be found in Appendix A 1). We have chosen this observable because it appears to be affected by large corrections to scaling, but the method can be applied to any universal quantity. From the discussion of Sec. II, we have for $\beta\rightarrow\beta_c$

$$g_4(\beta)=g_4^*+c_\Delta(\beta_c-\beta)^\Delta+\cdots, \tag{3.13}$$

where g_4^* is a universal constant, and c_Δ is a nonuniversal amplitude depending on the Hamiltonian. For improved models, as discussed before, $c_\Delta=0$. The traditional methods of analysis, e.g., those based on Padè (PA) and Dlog-Padè (DPA) approximants, are unable to handle an asymptotic behavior such as (3.13) unless Δ is an integer number, thus leading to a systematic error. Integral approximants allow for nonanalytic scaling corrections but, as already said, with the series of moderate length available today, they need to be biased to give correct results: without any bias they give estimates that are similar to those of PA's and DPA's [14]. At present the only analyses that are able to effectively take into account the confluent corrections use biased approximants, fixing the value of β_c and of the first nonanalytic exponent Δ (see, e.g., Refs. [10–12,14,15,36] for a discussion of this issue and for a presentation of the different methods used in the literature). The method we use has been proposed in Ref. [11] and generalized in Ref. [12]. The idea is to perform a Roskies transform (RT), i.e., the change of variables

$$z=1-(1-\beta/\beta_c)^\Delta, \tag{3.14}$$

so that the nonanalytic terms in $\beta_c-\beta$ become analytic in $1-z$. Therefore, the analysis of the resulting series by means of standard approximants should give correct results. For the models we are considering, the exponent Δ is approximately 1/2 [e.g., Ref. [20] reports $\Delta=0.532(6)$]; for simplicity, we have used the transformation (3.14) with $\Delta=1/2$.

We have analyzed the HT expansion of g_4 for the model with Hamiltonian (1.2) and $\lambda_6=0$ for several values of λ_4. We computed PA's, DPA's, and first-order integral (IA1) approximants of the series in β and of its RT in z. In Fig. 1 we plot the results as a function of λ_4^{-1}. The reported errors are related to the spread of the results obtained from the different approximants; see Appendix A 4 for details. The estimates obtained from the RT'ed series are independent of

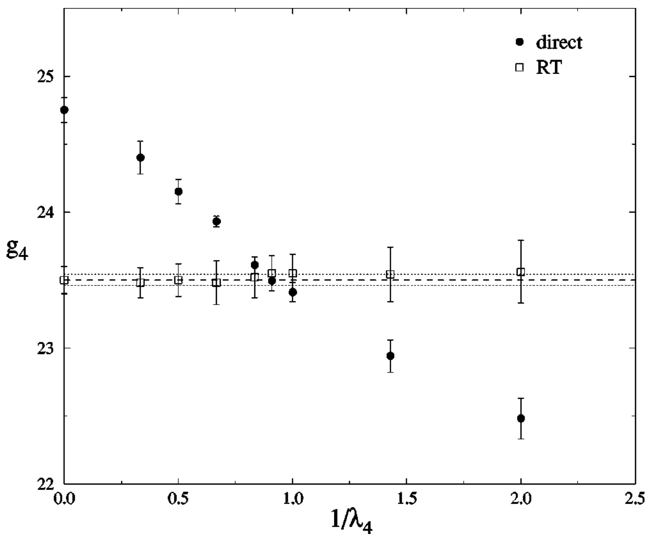

FIG. 1. Comparison of the determination of g_4 (plotted vs. $1/\lambda_4$) from HT series without (direct) and with the Roskies transform (RT) for the pure ϕ^4 lattice model. The dashed line marks the more precise estimate (with its error) we derived from the analysis of the IHT expansion.

λ_4 within error bars, giving the estimate $g_4^* \simeq 23.5$, in agreement with previous analyses of the HT expansion of the Ising model on various lattices using the RT or other types of biased approximants [14,15,36] (in Sec. V B we will improve this estimate by analyzing the IHT expansion). The independence of the result from the value of λ_4 clearly indicates that the RT is effectively able to take into account the nonanalytic behavior (3.13). On the other hand, the analysis of the series in β gives results that vary with λ_4 more than the spread of the approximants: for instance, the analysis of the series of the standard Ising model, corresponding to $\lambda_4 = \infty$, gives results that differ by more than 5% from the estimate quoted above, while the spread of the approximants is much smaller. Clearly there is a large systematic error. It is important to note that the direct analysis and the RT one coincide when $1.0 \lesssim \lambda_4 \lesssim 1.2$, i.e., in the region in which the leading nonanalytic corrections are small. This fact confirms our claim that the observed discrepancies are an effect of the confluent corrections.

The results presented above can be used to obtain an estimate of λ_4^* from the HT series alone: λ_4^* should fall in the interval in which the direct analysis gives results compatible with those obtained from the Roskies-transformed series. As we already mentioned, for $\lambda_6 = 0$ we obtain $\lambda_4^* = 1.1(1)$, while for $\lambda_6 = 1$ we get $\lambda_4^* = 1.9(1)$. The latter estimate was indeed the starting point of our Monte Carlo simulation. We have also tried to estimate λ_4^* in more direct ways, but all methods we tried were even less precise.

A similar method for the determination of the improved Hamiltonian from the HT series was presented in Ref. [8]. The optimal value of the parameter (called y in Ref. [8]) was determined comparing the results for the critical point $\beta_c(y)$ obtained using IA1's and DPA's: y^* is estimated from the value at which DPA and IA1 estimates of $\beta_c(y)$ agree between each other. It should be noticed that, for the double-Gaussian model, partial differential approximants and a later analysis of Nickel and Rehr [16] using a different method gave significantly different estimates of y^*.

IV. CRITICAL EXPONENTS

The analysis in Sec. III is encouraging and supports our basic assumption that the systematic error due to confluent singularities is largely reduced when analyzing IHT expansions. To further check this hypothesis we will compare results obtained from different improved Hamiltonians. This will provide an estimate of the remaining systematic error which is not covered by the spread of the results from different approximants.

The definition of the quantities we have considered and a detailed description of the method we used to generate and analyze the HT series is presented in Appendix A.

We computed β_c and γ from the analysis of the HT expansion of the magnetic susceptibility. We considered integral approximants of first, second, and third order. After a careful analysis we preferred the second-order integral approximants (IA2's), which turned out to be the most stable: most of the results we present in this section and in the related Appendix A 3 have been obtained by using IA2's. As a further check of the effectiveness of the approximants employed, we made use of the fact that χ (and ξ^2) must present an antiferromagnetic singularity at $\beta_c^{\rm af} = -\beta_c$ of the form [37]

$$\chi = c_0 + c_1(\beta - \beta_c^{\rm af})^{1-\alpha} + \cdots , \tag{4.1}$$

where α is the specific heat exponent, c_i are constants, and the ellipses represent higher-order singular or analytic corrections. We verified the existence of a singularity at $\beta \simeq -\beta_c$ in the approximants, and calculated the associated exponent. We also considered approximants that were biased by requiring the presence of two symmetric singularities at $\beta = \pm \beta_c$ [8]; the results obtained are consistent with the predicted behavior (4.1) (see Appendix A 3 and related Tables).

The exponent ν was obtained from the series of the second-moment correlation length

$$\xi^2 = \frac{m_2}{6\chi} \sim (\beta_c - \beta)^{2\nu}, \tag{4.2}$$

where m_i are the moments of the two-point function. We followed the procedure suggested in Ref. [38], i.e., we used the estimate of β_c obtained from χ to bias the analysis of ξ^2. For this purpose we used IA's biased by fixing β_c. We also considered approximants biased by forcing a pair of singularities at $\pm\beta_c$.

In Table III we report the results obtained for the Hamiltonians (1.2) with $\lambda_6 = 0$, $\lambda_4 = 1.10$ and with $\lambda_6 = 1$, $\lambda_4 = 1.90$, and for the Hamiltonian (1.4) with $D = 0.641$.

The errors are given as a sum of two terms: the first one is computed from the spread of the approximants; the second one is related to the uncertainty of the value of λ_4^* and D^*, and it is evaluated by changing λ_4 in the range $1.08 - 1.12$ for $\lambda_6 = 0$ and $1.86 - 1.94$ for $\lambda_6 = 1$, and D in the range $0.633 - 0.649$ for the spin-1 model. There is good agreement among the estimates of γ and ν obtained from the three improved Hamiltonians considered. This is an important check of our working hypothesis, i.e., that systematic errors

TABLE III. Our final estimates of γ, ν, η, and σ. The error is reported as a sum of two terms: the first one is related to the spread of the approximants; the second one is related to the uncertainty of the value of λ_4^* and D^*.

	γ	ν	η	σ
$\lambda_6=0$	1.23732(24+16)	0.63015(13+12)	0.0364(3+1)	0.0213(13+1)
$\lambda_6=1$	1.23712(26+31)	0.63003(13+23)	0.0363(3+2)	0.0213(14+2)
spin-1	1.23680(30+12)	0.62990(15+8)	0.0366(3+1)	0.0202(10+1)

due to confluent corrections are largely reduced. This will also be confirmed by the results for the universal ratios of amplitudes. We determine our final estimates by combining the results of the three improved Hamiltonians: the value is the weighted average of the three results, and the error is the smallest of the three errors. We obtain for γ and ν

$$\gamma = 1.2371(4), \tag{4.3}$$

$$\nu = 0.63002(23), \tag{4.4}$$

and by the hyperscaling relation $\alpha = 2 - 3\nu$

$$\alpha = 0.1099(7). \tag{4.5}$$

In Appendix A 3 we also report some further checks using the Monte Carlo estimate of β_c reported in Ref. [20] to bias the analysis of the series. The results are perfectly consistent. We mention that from the analysis of the antiferromagnetic singularity [cf. Eq. (4.1)], we obtain the estimate $\alpha = 0.105(10)$, which is consistent with result (4.5) obtained assuming hyperscaling.

From the results for γ and ν, we can obtain η by the scaling relation $\gamma = (2-\eta)\nu$. This gives $\eta = 0.0364(10)$, where the error is estimated by considering the errors on γ and ν as independent, which is of course not true. We can obtain an estimate of η with a smaller, yet reliable, error using the so-called critical-point renormalization method (CPRM) (see Ref. [9] and references therein). We obtain the results reported in Table III, with considerably smaller errors. Our final estimate is

$$\eta = 0.0364(4). \tag{4.6}$$

Moreover, using the scaling relations we obtain

$$\delta = \frac{5-\eta}{1+\eta} = 4.7893(22), \tag{4.7}$$

$$\beta = \frac{\nu}{2}(1+\eta) = 0.32648(18) \tag{4.8}$$

(the error on β has been estimated by considering the errors of ν and η as independent).

Finally, we consider the universal critical exponent, describing how the spatial anisotropy, which is present in physical systems with cubic symmetry (e.g., uniaxial magnets), vanishes when approaching the rotationally invariant fixed point [39]. For this class of systems the two-point function $G(x)$ is not rotationally invariant. Therefore, nonspherical moments are, in general, nonvanishing, but near the critical point they are depressed with respect to spherical moments carrying the same naive physical dimensions by a factor $\xi^{-\rho}$, where ρ is a universal critical exponent. From a field-theoretical point of view, space anisotropy is due to non-rotationally-invariant irrelevant operators in the effective Hamiltonian, whose presence depends essentially on the symmetries of the physical system, or of the lattice formulation. In Table III we report the results for $\sigma \equiv 2-\rho$ as obtained by analyses of the first nonspherical moments [cf. Eq. (A2)] using the CPRM. The exponent σ turns out to be very small:

$$\sigma = 0.0208(12), \tag{4.9}$$

and $\rho = 1.9792(12)$.

In Table IV we compare our results with some of the most recent estimates of the critical exponents γ, ν, η, α, and β. The table should give an overview of the state of the art for the various approaches.

Let us first note the good agreement of our IHT estimates with the very precise results of the recent Monte Carlo (MC) simulations of Refs. [17,19,20]. The small difference with the HT estimates of Ref. [13] (obtained from the standard Ising model) may be explained by the difficulty of controlling the effects of the confluent singularities, and by a systematic error induced by the uncertainty on the external input parameters (β_c and Δ) that are used in their biased analysis. The estimates of Refs. [5,7,8,16] have been obtained from a HT analysis of two families of models, the Klauder and the double-Gaussian models, on the bcc lattice. The results of these analyses are in good agreement with our IHT estimates, especially those by Nickel and Rehr [16]. The HT series for the double-Gaussian model were analyzed also in Ref. [8] where a higher estimate of γ was obtained. As pointed out in Ref. [16], the discrepancy is essentially due to the use of a higher estimate of the improvement parameter y^* with respect to that used in Ref. [16] (see the discussion at the end of Sec. III). Refs. [8,16] also report estimates of α obtained by analyzing the singularity of the susceptibility at the antiferromagnetic critical point. The result agrees with our estimate.

The agreement with the field-theoretical calculations is, overall, good. The slightly larger result for γ obtained in the analyses of Refs. [44,47] [using $O(g^7)$ series [46,50]] may be due to an underestimate of the systematic error due to the nonanalyticity of the Callan-Symanzik β function. Similar results have been obtained by Kleinert, who resummed the $O(g^7)$ expansion by a variational method [45], still neglecting confluent singularities at the infrared-stable fixed point. We shall return to this point later. A better agreement is found with the analysis of the $d=3$ g-expansion performed

TABLE IV. Theoretical estimates of critical exponents. See text for explanation of symbols in the first column. For values marked with an asterisk, the error is not quoted explicitly in the reference.

	γ	ν	η	α	β
IHT	1.2371(4)	0.63002(23)	0.0364(4)	0.1099(7)	0.32648(18)
HT (sc) [13]	1.2388(10)	0.6315(8)			
HT (bcc) [13]	1.2384(6)	0.6308(5)			
HT [16]	1.237(2)	0.6300(15)	0.0359(7)	0.11(2)	
HT [38]	1.239(3)	$0.632^{+0.002}_{-0.003}$			
HT [8]	1.2395(4)	0.632(1)		0.105(7)	
HT [7]	1.2378(6)	0.63115(30)			
HT [5]	1.2385(15)				
HT [4]	1.2385(25)	0.6305(15)			
MC [20]	1.2367(11)	0.6296(7)	0.0358(9)		
MC [17]		0.6298(5)	0.0366(8)		
MC [19]		0.6294(10)	0.0374(12)		
MC [40]		0.6308(10)			
MC [41]					0.3269(6)
MC [42]		0.625(1)	0.025(6)		
MC [43]	1.237(2)	0.6301(8)	0.037(3)	0.110(2)	0.3267(10)
ϵ exp$\vert_{\rm free}$ [44]	1.2355(50)	0.6290(25)	0.0360(50)		0.3257(25)
ϵ exp$\vert_{\rm BC}$ [44]	1.2380(50)	0.6305(25)	0.0365(50)		0.3265(15)
ϵ exp$\vert_{\rm BC}$ [14]	1.240(5)	0.631(3)			
$d=3$ g exp. [44]	1.2396(13)	0.6304(13)	0.0335(25)	0.109(4)	0.3258(14)
$d=3$ g exp. [45]	1.241*	0.6305*	0.0347(10)		
$d=3$ g exp. [46]	1.2378(6+18)	0.6301(5+11)	0.0355(9+6)		
$d=3$ g exp. [47]	1.2405(15)	0.6300(15)	0.032(3)		
ERG [48]		0.618(14)	0.054*		
ERG [49]	1.247*	0.638*	0.045*		

by Murray and Nickel, who allow for a more general nonanalytic behavior of the β function [46]. In Table IV, we quote the results of Ref. [46] with two erros: the first one is the resummation error, and the second one takes into account the uncertainty of g^*, which is estimated to be ~ 0.01. The results of the ϵ expansion were obtained from the $O(\epsilon^5)$ series calculated in Refs. [51,52]. We report estimates obtained by performing standard analyses (denoted as ''free'') and constrained analyses [53] (denoted by BC) that incorporate the knowledge of the exact two-dimensional values. Both are essentially consistent with our IHT estimates, but present a significantly larger uncertainty. In Table IV we also report the results obtained by approximately solving the exact renormalization-group equation (ERG); they seem to be much less precise. A more complete list of references pertaining to the theoretical determination of the critical exponents can be found in Ref. [44]. Concerning the exponent σ related to the rotational symmetry, the IHT results represent a substantial improvement of the estimates obtained by various approaches (HT and field theory) presented in Ref. [39].

Experimental results have been obtained by studying the liquid-vapor transition in simple fluids, and the different critical transitions in multicomponent fluid mixtures, uniaxial antiferromagnetic materials, and micellar systems. Many recent estimates can be found in Refs. [43,54,55]. In Table V we report some experimental results, most of them published after 1990. It is not a complete list of the published results, but it may be useful to get an overview of the experimental state of the art.

Even if the systems studied are quite different, the results substantially agree, although, examining them in greater detail, as already observed in Ref. [43], one can find small discrepancies. Moreover, they substantially agree with the theoretical predictions discussed above, confirming the fact that all these transitions are in the Ising universality class. It should also be noted that the experimental results are less accurate than the theoretical estimates.

V. THE EFFECTIVE POTENTIAL

A. Small-field expansion of the effective potential in the high-temperature phase

The effective potential (Helmholtz free energy) is related to the (Gibbs) free energy of the model. Indeed, if $M \equiv \langle \phi \rangle$ is the magnetization and H the magnetic field, one defines

$$\mathcal{F}(M) = MH - \frac{1}{V} \ln Z(H), \tag{5.1}$$

where $Z(H)$ is the partition function and the dependence on the temperature is always understood in the notation.

TABLE V. Experimental estimates of critical exponents. LV denotes the liquid-vapor transition in simple fluids, BM refers to a binary fluid mixture, MS to a uniaxial magnetic system, and MI to a micellar system.

	Ref.	γ	ν	η	α	β
LV	[56]				$0.1105^{+0.0250}_{-0.0270}$	
	[57]				0.1075(54)	
	[58]				0.1084(23)	
	[59]				0.111(1)	0.324(2)
	[60]					0.341(2)
	[61]			0.042(6)		
	[62]	1.233(10)				0.327(2)
BM	[63]				0.104(11)	
	[64]	1.09(3)				
	[65]	1.26(5)	0.64(2)			
	[66]	1.24(1)	0.606(18)		0.077(44)	0.319(14)
	[67]				0.105(8)	
	[68]					0.324(5), 0.329(2)
	[69]					0.329(4), 0.333(2)
	[70]		0.610(6)			
	[71]					0.336(30)
	[72]	1.228(39)	0.628(8)	0.0300(15)		
MS	[73]	1.25(2)	0.64(1)			
	[74]				0.115(4)	0.331(6)
	[75]				0.11(3)	
	[76]					0.325(2)
	[77]				0.11(3)	
	[78]	1.25(2)				0.315(15)
MI	[79]					0.34(8)
	[80]	1.18(3)	0.60(2)			
	[81]	1.216(13)	0.623(13)	0.039(4)		
	[82]	1.237(7)	0.630(12)			
	[83]	1.25(2)	0.63(1)			
	[84]	1.17(11)	0.65(4)			

The global minimum of the effective potential determines the value of the order parameter which characterizes the phase of the model. In the high-temperature or symmetric phase, the minimum is unique and located at $M=0$. According to the Ginzburg-Landau theory, as the temperature decreases below the critical value, the effective potential takes a double-well shape. The order parameter does not vanish anymore and the system is in the low-temperature or broken phase. Actually, in the broken phase the double-well shape is not correct because the effective potential must be convex [85]. In this phase it should present a flat region around the origin.

In the high-temperature phase the effective potential admits an expansion around $M=0$:

$$\Delta\mathcal{F}\equiv\mathcal{F}(M)-\mathcal{F}(0)=\sum_{j=1}^{\infty}\frac{1}{(2j)!}a_{2j}M^{2j}. \tag{5.2}$$

The coefficients a_{2j} can be expressed in terms of renormalization-group invariant quantities. Introducing a renormalized magnetization

$$\varphi^2=\frac{\xi(t,H=0)^2M(t,H)^2}{\chi(t,H=0)}, \tag{5.3}$$

where t is the reduced temperature, one may write

$$\Delta\mathcal{F}=\frac{1}{2}m^2\varphi^2+\sum_{j=2}m^{d-j(d-2)}\frac{1}{(2j)!}g_{2j}\varphi^{2j}. \tag{5.4}$$

Here $m=1/\xi$, g_{2j} are functions of t only, and d is the space dimension. In field theory φ is the expectation value of the zero-momentum renormalized field. For $t\rightarrow 0$ the quantities g_{2j} approach universal constants (which we indicate with the same symbol) that represent the zero-momentum $2j$-point renormalized coupling constants. By performing a further rescaling

$$\varphi=\frac{m^{(d-2)/2}}{\sqrt{g_4}}z \tag{5.5}$$

in Eq. (5.4), the free energy can be written as

$$\Delta\mathcal{F}=\frac{m^d}{g_4}A(z), \tag{5.6}$$

where

$$A(z)=\frac{1}{2}z^2+\frac{1}{4!}z^4+\sum_{j=3}\frac{1}{(2j)!}r_{2j}z^{2j}, \tag{5.7}$$

and

$$r_{2j}=\frac{g_{2j}}{g_4^{j-1}} \quad j\geq 3. \tag{5.8}$$

One can show that $z\propto t^{-\beta}M$, and that the equation of state can be written in the form

$$H\propto t^{\beta\delta}\frac{\partial A(z)}{\partial z}. \tag{5.9}$$

The effective potential $\mathcal{F}(M)$ admits a power-series expansion also near the coexistence curve, i.e., for $t<0$ and $H=0$. If $M_0=\lim_{H\to 0^+}M(H)$, for $M>M_0$ (i.e., for $H\geq 0$) we have

$$\delta\mathcal{F}\equiv\mathcal{F}(M)-\mathcal{F}(M_0)=\sum_{j=2}\frac{1}{j!}a_j(M-M_0)^j. \tag{5.10}$$

In terms of the renormalized magnetization φ we can rewrite

$$\delta\mathcal{F}=\frac{1}{2}m^2(\varphi-\varphi_0)^2+\sum_{j=3}m^{d-j(d-2)/2}\frac{1}{j!}g_j^-(\varphi-\varphi_0)^j, \tag{5.11}$$

where $m\equiv 1/\xi^-$ and ξ^- is the second-moment correlation length defined in the low-temperature phase. For $t\to 0^-$, the quantities g_j^- approach universal constants that represent the low-temperature zero-momentum j-point renormalized coupling constants. A simpler parametrization can be obtained if we introduce [86]

$$u\equiv\frac{M}{M_0}, \tag{5.12}$$

so that

$$\delta\mathcal{F}=\frac{m^d}{w^2}B(u), \tag{5.13}$$

where

$$w^2\equiv\lim_{T\to T_c^-}\lim_{H\to 0}\frac{\chi}{M^2\xi^d}. \tag{5.14}$$

The scaling function $B(u)$ has the following expansion:

$$B(u)=\frac{1}{2}(u-1)^2+\sum_{j=3}\frac{1}{j!}v_j(u-1)^j, \tag{5.15}$$

where

$$v_j=\frac{g_j^-}{w^{j-2}}. \tag{5.16}$$

B. Four-point zero-momentum renormalized coupling

The four-point coupling $g\equiv g_4$ plays an important role in the field-theoretic perturbative expansion at fixed dimension [87], which provides an accurate description of the critical region in the symmetric phase. In this approach, any universal quantity is obtained from a series in powers of g (g expansion), which is then resummed and evaluated at the fixed-point value of g, g^* (see, e.g., Refs. [47,50]). The theory is renormalized at zero momentum by requiring

$$\Gamma^{(2)}(p)=Z^{-1}[M^2+p^2+O(p^4)], \tag{5.17}$$

$$\Gamma^{(4)}(0,0,0,0)=Z^{-2}Mg. \tag{5.18}$$

When $M\to 0$ the coupling g is driven toward an infrared-stable zero g^* of the corresponding Callan-Symanzik β function

$$\beta(g)\equiv M\left.\frac{\partial g}{\partial M}\right|_{g_0,\Lambda}. \tag{5.19}$$

In this context a rescaled coupling is usually introduced (see, e.g., Ref. [1]):

$$\bar{g}=\frac{3}{16\pi}g. \tag{5.20}$$

An important issue in this field-theoretical approach concerns the analytic properties of $\beta(g)$, which are relevant for the resummation of the g expansion. General renormalization-group arguments predict a nonanalytic behavior of $\beta(g)$ at $g=g^*$ [87]. One expects a behavior of the form [2,88]

$$\beta(g)=-\omega(g^*-g)+b_1(g^*-g)^2+\cdots+c_1(g^*-g)^{1+1/\Delta}+\cdots+d_1(g^*-g)^{\Delta_2/\Delta}+\cdots \tag{5.21}$$

($\Delta=\omega\nu$ and Δ_2 are scaling correction exponents). In the framework of the $1/N$ expansion of O(N) ϕ^4 models, the analysis [14] of the next-to-leading order of the Callan-Symanzik β function, calculated in Ref. [89], shows explicitly the presence of confluent singularities of the form (5.21).

In the fixed-dimension field-theoretical approach, a precise determination of g^* is crucial, since the critical exponents are obtained by evaluating appropriate (resummed) anomalous dimensions at g^*. The resummation of the g expansion is usually performed following the Le Guillou-Zinn-Justin (LZ) procedure [47], which assumes the analyticity of the β function. The presence of confluent singularities may cause a slow convergence to the correct fixed-point value, leading to an underestimate of the uncertainty derived from stability criteria.

We have computed $g^*\equiv g_4^*$ from our IHT series by calculating the critical limit of the quantity g_4 defined in Eq. (3.12). A description of our analysis can be found in Appendix A 4. The results are reported in Table VI. We find good agreement among the results of the three improved Hamiltonians, which lead to our final estimate:

$$g^*=23.49(4), \quad \bar{g}^*=1.402(2). \tag{5.22}$$

TABLE VI. Results for g_4^*, r_6, r_8, r_{10}, c_2, and c_3 derived from the analysis of the IHT series (see Appendix A). The error is reported as a sum of two terms: the first one is related to the spread of the approximants; the second one is related to the uncertainty of the value of λ_4^* and D^*.

	g_4^*	r_6	r_8	r_{10}	$10^4 c_2$	$10^4 c_3$
$\lambda_6=0$	23.499(16+20)	2.051(7+2)	2.23(5+4)	−14(4)	−3.582(7+6)	0.085(6)
$\lambda_6=1$	23.491(21+40)	2.050(5+4)	2.23(5+6)	−13(5)	−3.574(7+20)	0.086(4)
spin-1	23.487(18+20)	2.046(2+3)	2.34(5+3)	−8(25)	−3.568(11+4)	0.090(4)

Table VII presents a selection of estimates of $\bar{g}^*$ obtained by different approaches.

The HT estimates of Refs. [14,15,36] were obtained by using the RT or appropriate biased approximants in order to handle the leading confluent correction. The larger result of Ref. [96] could be explained by an effect of the scaling corrections. Field-theoretical estimates are reasonably consistent, especially those obtained from a constrained analysis of the $O(\epsilon^4)$ ϵ expansion [14]. In the $d=3$ g-expansion approach, $\bar{g}^*$ is determined from the zero of $\beta(g)$ after resumming its available $O(g^7)$ series. The results obtained using the LZ resummation method [44] show a slight discrepancy from our IHT estimates. This difference can explain the apparent discrepancy found in the determination of γ. Indeed, the sensitivity of γ to $\bar{g}^*$, quantified in Ref. [44] through $d\gamma/d\bar{g}^*\simeq 0.18$, tells us that changing the value of $\bar{g}^*$ from 1.411 [which is the value obtained from the zero of $\beta(g)$] to 1.402 shifts γ from 1.2396 to 1.2380, which is much closer to the IHT estimate $\gamma=1.2371(4)$. Similarly for ν, using $d\nu/d\bar{g}^*\simeq 0.11$ [44], ν would change from 0.6304 to 0.6294, which is quite acceptable, since a residual uncertainty due to the resummation of $\nu(g)$ is still present. The more general analysis of the g expansion of Ref. [46] leads to a smaller value $\bar{g}^*=1.40$, with an uncertainty estimated by the authors to be about 1%. In Table VII we also report estimates obtained by approximately solving the exact renormalization group equation [49,48] (ERG), and from a dimensional expansion of the Green's functions around $d=0$ [91] (d exp.). Concerning Monte Carlo (MC) results, we mention that the result of Ref. [90] has been obtained by studying the probability distribution of the average magnetization (see also Ref. [98] for a work employing a similar approach). The other estimates have been obtained from fits to data in the neighborhood of β_c. In Ref. [18] Monte Carlo simulations were performed using the Hamiltonian (1.2) with $\lambda_6=0$ and $\lambda_4=1$, which is close to its optimal value. A fit to the data of g_4, kindly made available to us by the authors, gives the estimate $g^*=23.41(24)$ [i.e., $\bar{g}^*=1.397(14)$], which is in agreement with our IHT estimate. In Ref. [92] a finite-size scaling technique is used to obtain data for large correlation lengths, then the estimate of g_4^* is extracted by a fit taking into account the leading scaling correction. The Monte Carlo estimates of Refs. [94,95] were larger because the effects of scaling corrections were neglected, as already observed in Ref. [14]. A more complete list of references regarding this issue can be found in Ref. [14].

C. Higher-order zero-momentum renormalized couplings

To compute the HT series of the effective-potential parameters r_{2j} defined in Eq. (5.8), we rewrite them in terms of the zero-momentum connected $2j$-point Green's functions χ_{2j} as

$$r_6=10-\frac{\chi_6\chi_2}{\chi_4^2}, \tag{5.23}$$

$$r_8=280-56\frac{\chi_6\chi_2}{\chi_4^2}+\frac{\chi_8\chi_2^2}{\chi_4^3}, \tag{5.24}$$

$$r_{10}=15400-4620\frac{\chi_6\chi_2}{\chi_4^2}+126\frac{\chi_6^2\chi_2^2}{\chi_4^4}+120\frac{\chi_8\chi_2^2}{\chi_4^3}-\frac{\chi_{10}\chi_2^3}{\chi_4^4}, \tag{5.25}$$

etc. Details of the analysis of the series are reported in Appendix A 4. Combining the results reported in Table VI, we obtain the following estimates:

$$r_6=2.048(5), \tag{5.26}$$

$$r_8=2.28(8), \tag{5.27}$$

$$r_{10}=-13(4). \tag{5.28}$$

TABLE VII. Estimates of $\bar{g}^*\equiv 3g^*/(16\pi)$. (sc) and (bcc) in the HT estimates of Ref. [15] denote simple cubic and bcc lattice, respectively. For values marked with an asterisk, the error is not quoted explicitly in the reference.

IHT	HT	ϵ exp.	$d=3$ g exp.	MC	d exp.	ERG
1.402(2)	1.408(7) (sc) [15]	1.397(8) [14]	1.411(4) [44]	1.39(3) [90]	1.412(14) [91]	1.23(21) [48]
	1.407(6) (bcc) [15]	1.391* [44]	1.40* [46]	1.408(12) [92]		1.72* [49]
	1.406(9) [14]		1.415* [93]	1.49(3) [94]		
	1.414(6) [36]		1.416(5) [47]	1.462(12) [95]		
	1.459(9) [96]					
	1.42(9) [97]					

TABLE VIII. Estimates of r_{2j}. For the references reporting only estimates of g_{2j} (see Refs. [90,95,97]), the errors we quote for r_{2j} have been calculated by considering the estimates of g_{2j} as uncorrelated. For values marked with an asterisk, the error is not quoted explicitly in the reference.

	IHT	HT	ϵ exp.	$d=3$ g exp.	MC	ERG
r_6	2.048(5)	1.99(6) [36] 2.157(18) [96] 2.25(9) [100] 2.5(5) [97]	2.058(11) [99] 2.12(12) [44]	2.053(8) [44] 2.060* [93]	2.72(23) [90] 3.37(11) [92] 3.26(26) [95]	2.064(36) [48] 1.92* [49]
r_8	2.28(8)	2.7(4) [36]	2.48(28) [99] 2.42(30) [44]	2.47(25) [44]		2.47(5) [48] 2.18* [49]
r_{10}	−13(4)	−4(2) [36]	−20(15) [99] −12.0(1.1) [44]	−25(18) [44]		−18(4) [48]

From the results for r_{2j} we can obtain estimates of the couplings

$$g_6=g^2r_6=1130(5), \tag{5.29}$$

$$g_8=g^3r_8=2.96(11)\times 10^4, \tag{5.30}$$

$$g_{10}=g^4r_{10}=-4.0(1.2)\times 10^6. \tag{5.31}$$

In the literature several approaches have been used for the determination of the couplings g_{2j}. Table VIII presents a review of the available estimates of r_6, r_8, and r_{10}.

We also mention the estimate $r_{10}=-10(2)$ that we will obtain in Sec. VII by studying the equation of state. The agreement with the field-theoretic calculations based on the ϵ expansion [44,99] and on the $d=3$ g expansion [44] is good. Precise estimates of r_{2j} have also been obtained in Ref. [48] (see also Ref. [49]) by ERG, although the estimate of g_4^* by the same method is not as good. Additional results have been obtained from HT expansions [36,96,97] and Monte Carlo simulations [90,95] of the Ising model. The Monte Carlo results do not agree with the results of other approaches, especially those of Refs. [92,95], which are obtained using finite-size scaling techniques. But one should consider the difficulty of such calculations due to the subtractions that must be performed to compute the irreducible correlation functions. A more complete list of references regarding this issue can be found in Refs. [23,44,99].

VI. THE TWO-POINT FUNCTION

The critical behavior of the two-point correlation function $G(x)$ of the order parameter is relevant to the description of critical scattering phenomena, which can be observed in many experiments, such as light and x-ray scattering in fluids, magnets, etc. In the Born approximation the cross section Γ_{fi} for particles of incoming momentum p_i and outgoing momentum p_f is proportional to the component $k=p_f-p_i$ of the Fourier transform of $G(x)$:

$$\Gamma_{fi}\propto \widetilde{G}(p_f-p_i). \tag{6.1}$$

As a consequence of the critical behavior of the two-point function $G(x)$ at T_c,

$$\widetilde{G}(k)\sim \frac{1}{k^{2-\eta}}, \tag{6.2}$$

the cross section for $k\rightarrow 0$ (forward scattering) diverges as $T\rightarrow T_c$. When strictly at criticality, Eq. (6.2) holds for all $k\ll\Lambda$, where Λ is a generic cutoff related to the microscopic structure of the statistical system, e.g., the inverse lattice spacing in the case of lattice models. In the vicinity of the critical point, where the relevant correlation length ξ is large but finite, the behavior (6.2) occurs for $\Lambda\gg k\gg 1/\xi$. At low momentum, $k\ll 1/\xi$, experiments show that $G(x)$ is well approximated by a Gaussian (Ornstein-Zernike) behavior,

$$\frac{\widetilde{G}(0)}{\widetilde{G}(k)}\simeq 1+\frac{k^2}{M^2}, \tag{6.3}$$

where $M\sim 1/\xi$ is a mass scale defined at zero momentum (for a general discussion, see, e.g., Ref. [101]). Corrections to Eq. (6.3) are present, and reflect, once more, the non-Gaussian nature of the Wilson-Fisher fixed point. The above-mentioned experimental observations, confirmed by theoretical studies [39,102], show that they are small. In the following we will improve the determination of the critical two-point function at low momentum using IHT series.

In order to study the low-momentum universal critical behavior of the two-point function $G(x)=\langle\phi(x)\phi(0)\rangle$, we consider the scaling function

$$g(y)=\chi/\widetilde{G}(k), \quad y\equiv k^2/M^2, \tag{6.4}$$

($M\equiv 1/\xi$ and ξ is the second-moment correlation length) in the critical limit $k,M\rightarrow 0$ with y fixed. The scaling function $g(y)$ can be expanded in powers of y around $y=0$:

$$g(y)=1+y+\sum_{i=2}^{\infty} c_i y^i. \tag{6.5}$$

Other important quantities which characterize the low-momentum behavior of $g(y)$ are the critical limit of the ratios

$$S_M\equiv M_{\rm gap}^2/M^2, \tag{6.6}$$

$$S_Z\equiv \chi M^2/Z_{\rm gap}, \tag{6.7}$$

TABLE IX. c_i and S_M-1 obtained from $O(\epsilon^3)$ series: unconstrained analysis (unc.) and analyses constrained in dimensions $d=1,2$.

	Unc.	$d=1$	$d=2$	$d=1,2$
$10^4(S_M-1)$	$-4.4(1.0)$	$-3.3(8)$	$-3.3(5)$	$-3.24(36)$
$10^4 c_2$	$-4.3(9)$	$-3.2(8)$	$-3.3(4)$	$-3.30(21)$
$10^5 c_3$	1.13(27)	0.84(22)	0.76(17)	0.69(10)
$10^6 c_4$	$-0.50(13)$	$-0.37(10)$	$-0.32(8)$	$-0.27(5)$

where $M_{\rm gap}$ (the mass gap of the theory) and $Z_{\rm gap}$ determine the long-distance behavior of the two-point function:

$$G(x)\approx \frac{Z_{\rm gap}}{4\pi|x|}e^{-M_{\rm gap}|x|}. \tag{6.8}$$

The critical limits of S_M and S_Z are related to the negative zero y_0 of $g(y)$ closest to the origin by

$$S_M=-y_0, \tag{6.9}$$

$$S_Z=\left.\frac{\partial g(y)}{\partial y}\right|_{y=y_0}. \tag{6.10}$$

The coefficients c_i can be related to the critical limit of appropriate dimensionless ratios of spherical moments of $G(x)$ (as shown explicitly in Appendix A 1) and can be calculated by analyzing the corresponding HT series. Some details of the analysis of our HT series are reported in Appendix A 4. In Table VI we report the results for $\lambda_6=0,1$ and the spin-1 model. We obtain the estimates

$$c_2=-3.576(13)\times 10^{-4}, \tag{6.11}$$

$$c_3=0.87(4)\times 10^{-5}, \tag{6.12}$$

and the bound

$$-10^{-6}\lesssim c_4<0. \tag{6.13}$$

The constants c_i and S_M can also be calculated by field-theoretic methods. They have been computed to $O(\epsilon^3)$ in the framework of the ϵ expansion [103], and to $O(g^4)$ in the framework of the $d=3$ g expansion [39]. In Table IX we report the results of constrained analyses of the $O(\epsilon^3)$ ϵ expansion of c_i and S_M-1, using exact results in $d=2,1$ ($S_M=1$ and $c_i=0$ in $d=1$; two-dimensional values will be reported in Table XI) and following the method of Ref. [14].

Since the constants c_i are of order $O(\epsilon^2)$, we analyzed the $O(\epsilon)$ series for c_i/ϵ^2. Errors are indicative, since the series are short. In Table X we compare the estimates obtained by various approaches: they all agree within the quoted errors.

As already observed in Ref. [39], the coefficients show the pattern

$$c_i \ll c_{i-1} \ll \cdots \ll c_2 \ll 1 \quad \text{for} \quad i\geqslant 3. \tag{6.14}$$

Therefore, a few terms of the expansion of $g(y)$ in powers of y should be a good approximation in a relatively large region around $y=0$, larger than $|y|\lesssim 1$. This is in agreement with the theoretical expectation that the singularity of $g(y)$ nearest to the origin is the three-particle cut [104,103]. If this is the case, the convergence radius r_g of the Taylor expansion of $g(y)$ is $r_g=9S_M$. Since, as we shall see, $S_M\simeq 1$, at least asymptotically we should have

$$c_{i+1}\simeq \frac{1}{9}c_i. \tag{6.15}$$

This behavior can be checked explicitly in the large-N limit of the N-vector model [39]. In two dimensions, the critical two-point function can be written in terms of the solutions of a Painlevé differential equation [105] and it can be verified explicitly that $r_g=9S_M$. In Table XI we report the values of S_M and c_i for the two-dimensional Ising model.

Assuming the pattern (6.14), we may estimate S_M and S_Z from c_2, c_3, and c_4. Indeed from the equation $g(y_0)=0$, where $y_0=-S_M$, we obtain

$$S_M=1+c_2-c_3+c_4+2c_2^2+\cdots, \tag{6.16}$$

$$S_Z=1-2c_2+3c_3-4c_4-2c_2^2+\cdots, \tag{6.17}$$

where the ellipses indicate contributions that are negligible with respect to c_4. In Ref. [39] the relation (6.16) has been confirmed by a direct analysis of the HT series of S_M. From Eqs. (6.16) and (6.17) we obtain $S_M=0.999634(4)$ [from

TABLE X. Estimates of S_M and c_i. (sc) and (bcc) denote the simple cubic and the body-centered-cubic lattice, respectively.

	IHT	HT	ϵ exp.	$d=3$ g exp.
c_2	$-3.576(13)\times 10^{-4}$	$-3.0(2)\times 10^{-4}$ [39] $-5.5(1.5)\times 10^{-4}$ (sc) [102] $-7.1(1.5)\times 10^{-4}$ (bcc) [102]	$-3.3(2)\times 10^{-4}$	$-4.0(5)\times 10^{-4}$
c_3	$0.87(4)\times 10^{-5}$	$1.0(1)\times 10^{-5}$ [39] $0.5(2)\times 10^{-5}$ (sc) [102] $0.9(3)\times 10^{-5}$ (bcc) [102]	$0.7(1)\times 10^{-5}$	$1.3(3)\times 10^{-5}$
c_4	$-10^{-6}\lesssim c_4<0$		$-0.3(1)\times 10^{-6}$	$-0.6(2)\times 10^{-6}$
S_M	0.999634(4)	0.99975(10) [39]	0.99968(4)	0.99959(6)

TABLE XI. Values of S_M and c_i for the two-dimensional Ising model in the high- and low-temperature phases.

High temperature [106]	Low temperature
$S_M = 0.999196337056$	$S_M^- = 0.399623590999$
$c_2 = -0.7936796064\times 10^{-3}$	$c_2^- = -0.42989191603$
$c_3 = 0.109599108\times 10^{-4}$	$c_3^- = 0.5256121845$
$c_4 = -0.3127446\times 10^{-6}$	$c_4^- = -0.8154613925$
$c_5 = 0.126670 \times 10^{-7}$	$c_5^- = 1.422603449$
$c_6 = -0.62997\times 10^{-9}$	$c_6^- = -2.663354573$

which we can derive an estimate of the ratio $Q_\xi^+ \equiv f_{\rm gap}^+/f^+ = 1.000183(2)$, cf. Eqs. (B5) and (B6)] and $S_Z = 1.000741(7)$.

We can also use our results to improve the phenomenological model proposed by Bray [103]. If we parametrize the large-y behavior of $g(y)$ as [107]

$$g(y)^{-1} = \frac{A_1}{y^{1-\eta/2}}\left(1 + \frac{A_2}{y^{(1-\alpha)/(2\nu)}} + \frac{A_3}{y^{1/(2\nu)}}\right), \quad (6.18)$$

then, by using our estimates of the critical exponents and the phenomenological function of Ref. [103], we obtain the following values for the coefficients:

$$A_1 \approx 0.918, \quad A_2 \approx 2.55, \quad A_3 \approx -3.45. \quad (6.19)$$

Estimating reliable errors on these results is practically impossible, since it is difficult to assess the systematic error due to the many uncontrolled simplifications that are used. It is, however, reassuring that they are in reasonable agreement the ϵ-expansion predictions [103]

$$A_1 \approx 0.92, \quad A_2 \approx 1.8, \quad A_3 \approx -2.7, \quad (6.20)$$

and with the results of a recent experimental study [61]

$$A_1 = 0.915(21), \quad A_2 = 2.05(80), \quad A_3 = -2.95(80). \quad (6.21)$$

Bray's phenomenological expression also makes predictions for the coefficients c_i. The pattern (6.15) is built into the approach. We find $c_2 = -4.2\times 10^{-4}$ and $c_3 = 1.0\times 10^{-5}$, in good agreement with our IHT estimates. Therefore, Bray's expression provides a good description of $g(y)$ for small and large values of y. However, in the intermediate crossover region, as already observed in Ref. [61], the agreement is worse: Bray's interpolation is lower by 20–50 % than the experimental result.

In the low-temperature phase, for $y\to 0$, the two-point function also admits a regular expansion of the form (6.5). However, the deviation from the Gaussian behavior is much larger. The leading coefficient c_2^- is larger than c_2 by about two orders of magnitude [108]. Moreover, by analyzing the low-temperature series published in Ref. [109] one gets $S_M^- = 0.938(8)$ [and, correspondingly, $Q_\xi^- \equiv f_{\rm gap}^-/f^- = 1.032(4)$]. Thus S_M^- shows a much larger deviation from one (the Gaussian value) than the corresponding high-temperature phase quantity S_M. The two-dimensional Ising model shows even larger deviations from Eq. (6.3), as one can see from the values of S_M^- and c_i^- reported in Table XI. Note that in the low-temperature phase of the two-dimensional Ising model, the singularity at $k^2 = -M_{\rm gap}^2$ of $\widetilde{G}(k)$ is not a simple pole, but a branch point [105]. As a consequence, $g(y)$ is not analytic for $|y| > S_M^-$, and therefore the convergence radius of the expansion around $y=0$ is S_M^-. For discussions of the analytic structure of $g(y)$ in the low-temperature phase of the three-dimensional Ising model, see, e.g., Refs. [108,103,104,110].

VII. THE CRITICAL EQUATION OF STATE

A. Parametric representation

The critical equation of state provides relations among the thermodynamical quantities in the neighborhood of the critical temperature, in both phases. From this equation one can then derive all the universal ratios of amplitudes involving quantities defined at zero-momentum (i.e., integrated in the volume), such as specific heat, magnetic susceptibility, etc.

From the analysis of IHT series we have obtained the first few nontrivial terms of the small-field expansion of the effective potential in the high-temperature phase. This provides corresponding information for the equation of state

$$H \propto t^{\beta\delta} F(z), \quad (7.1)$$

where $z \propto M t^{-\beta}$ and, using Eq. (5.9),

$$F(z) = \frac{\partial A(z)}{\partial z} = z + \tfrac{1}{6} z^3 + \sum_{m=2} F_{2m+1} z^{2m+1} \quad (7.2)$$

with

$$F_{2m-1} = \frac{1}{(2m-1)!} r_{2m}. \quad (7.3)$$

The function $H(M,t)$ representing the external field in the critical equation of state (7.1) satisfies Griffith's analyticity: it is regular at $M=0$ for $t>0$ fixed and at $t=0$ for $M>0$ fixed. The first region corresponds to small z in Eq. (7.1), while the second is related to large z, where $F(z)$ can be expanded in the form

$$F(z) = z^\delta \sum_{n=0} F_n^\infty z^{-n/\beta}. \quad (7.4)$$

Of course, F_n^∞ are universal constants.

To reach the coexistence curve, i.e., $t<0$ and $H=0$, one should perform an analytic continuation in the complex t plane [1,23]. The spontaneous magnetization is related to the complex zero z_0 of $F(z)$. Therefore, the description of the coexistence curve is related to the behavior of $F(z)$ in the neighborhood of z_0. In order to obtain a representation of the critical equation of state that is valid in the whole critical region, one may use parametric representations, which implement in a simple way all scaling and analytic properties. One parametrizes M and t in terms of R and θ [24–26]:

$$M = m_0 R^\beta \theta,$$

$$t = R(1-\theta^2), \quad (7.5)$$

$$H = h_0 R^{\beta\delta} h(\theta),$$

where h_0 and m_0 are normalization constants. The function $h(\theta)$ is odd and regular at $\theta=1$ and at $\theta=0$. The constant h_0 can be chosen so that $h(\theta)=\theta+O(\theta^3)$. The zero of $h(\theta)$, $\theta_0>1$, represents the coexistence curve $H=0$, $T<T_c$. The parametric representation satisfies the requirements of regularity of the equation of state. One expects at most an essential singularity on the coexistence curve [111].

The relation between $h(\theta)$ and $F(z)$ is given by

$$z = \rho\theta(1-\theta^2)^{-\beta}, \tag{7.6}$$

$$h(\theta) = \rho^{-1}(1-\theta^2)^{\beta\delta} F(z(\theta)), \tag{7.7}$$

$\theta>0$, and hyperscaling implies that $\beta\delta=\beta+\gamma$. Note that this mapping is invertible only in the region $\theta<\theta_l$, where $\theta_l=(1-2\beta)^{-1/2}$ is the solution of the equation $z'(\theta)=0$. Thus the values of θ that are relevant for the critical equation of state, i.e., $0\leqslant\theta\leqslant\theta_0$, must be smaller than θ_l. This fact will not be a real limitation for us, since the range of values of θ involved in our calculations (which will be $0\leqslant\theta^2\leqslant\theta_0^2 \lesssim 1.40$) will always be far from the limiting value $\theta_l^2 \simeq 2.88$.

As a consequence of Eqs. (7.5), (7.6), and (7.7), we easily obtain the relationships

$$\frac{M}{t^\beta} = \left(\frac{m_0}{\rho}\right) z, \quad \frac{H}{t^{\beta\delta}} = \left(\frac{h_0}{\rho}\right) F(z). \tag{7.8}$$

We can therefore treat ρ as a free parameter, and the scaling relations between physical variables will not depend on ρ, provided that m_0 and h_0 are rescaled with ρ. In the exact parametric equation, the value of ρ may be chosen arbitrarily but, as we shall see, when adopting an approximation procedure the dependence on ρ is not eliminated, and it may become important to choose the value of this parameter properly in order to optimize the approximation.

From θ_0 one can obtain the universal rescaled spontaneous magnetization [23], i.e., the complex zero z_0 of $F(z)$,

$$z_0 = |z_0| e^{-i\pi\beta}, \quad |z_0| = \rho\theta_0(\theta_0^2-1)^{-\beta}. \tag{7.9}$$

From the function $h(\theta)$ one can calculate the universal ratios of amplitudes. In Appendix B we report the definitions of the universal ratios of amplitudes that have been introduced in the literature, and the corresponding expressions in terms of $h(\theta)$.

Expanding $h(\theta)$ in (odd) powers of θ,

$$h(\theta) = \theta + \sum_{n=1} h_{2n+1}\theta^{2n+1}, \tag{7.10}$$

and using Eq. (7.7), one can find the relations among h_{2n+1} and the coefficients F_{2m+1} of the expansion of $F(z)$. The procedure is explained in Appendix C, and the general result is

$$h_{2n+1} = \sum_{m=0}^{n} c_{n,m}\rho^{2m} F_{2m+1}, \tag{7.11}$$

where

$$c_{n,m} = \frac{1}{(n-m)!}\prod_{k=1}^{n-m}(2\beta m-\gamma+k-1); \tag{7.12}$$

note that $c_{n,n}=1$. In general, h_{2n+1} depends on γ, β, and on the coefficients F_{2m+1} with $m\leqslant n$.

We shall need the explicit form of the first two coefficients:

$$h_3 = \tfrac{1}{6}\rho^2 - \gamma, \tag{7.13}$$

$$h_5 = \tfrac{1}{2}\gamma(\gamma-1) + \tfrac{1}{6}(2\beta-\gamma)\rho^2 + F_5\rho^4. \tag{7.14}$$

B. Approximation scheme based on stationarity

In Ref. [23], Guida and Zinn-Justin use the first few coefficients of the small-z expansion of $F(z)$ to get polynomial approximations of $h(\theta)$ that should provide a description that is reliable in the whole critical region. The approximations considered are truncations of the small-θ expansion of $h(\rho,\theta)$, i.e.,

$$h^{(t)}(\rho,\theta) = \theta + \sum_{n=1}^{t-1} h_{2n+1}(\rho)\theta^{2n+1}, \tag{7.15}$$

where $h_{2n+1}(\rho)$ are given by Eq. (7.11). We follow a similar strategy, with a significant difference in the procedure adopted in order to fix the value of ρ.

By Eqs. (7.11) and (7.12), the coefficients $h_{2n+1}(\rho)$ included in Eq. (7.15) are written in terms of the t parameters γ, β, $F_5 \cdots F_{2t-1}$. In practice, only the first coefficients of the small-θ expansion of $h(\theta)$ are well determined, since we have good estimates only for the first few F_{2m+1}. Once the order of the truncation has been decided, one may exploit the freedom of choosing ρ to optimize the approximation of $h(\theta)$. In this way one may hope to obtain a good approximation even for small values of t. Reference [23] proposes to determine the optimal value of ρ by minimizing the absolute value of $h_{2t-1}(\rho)$, i.e., the coefficient of the highest-order term considered. The idea underlying this procedure is to increase the importance of small powers of θ. Our approach is different.

Our starting point is the independence from ρ of the scaling function $F(z)$ and, as a consequence, of all universal ratios of amplitudes that can be extracted from it. Of course, this property does not hold anymore when we start from a truncated function $h^{(t)}(\rho,\theta)$, i.e., if we compute universal quantities from a function $F^{(t)}(\rho,z)$ defined by

$$F^{(t)}(\rho,z) \equiv \tilde{F}^{(t)}(\rho,\theta(\rho,z)), \tag{7.16}$$

where

$$\tilde{F}^{(t)}(\rho,\theta) = \frac{\rho h^{(t)}(\rho,\theta)}{(1-\theta^2)^{\beta\delta}} \tag{7.17}$$

and $\theta(\rho,z)$ is obtained by inverting Eq. (7.6).

In order to optimize ρ for a given truncation $h^{(t)}(\rho,\theta)$, we propose a procedure based on the physical requirement of minimal dependence on ρ of the resulting universal function

$F^{(t)}(\rho,z)$. This can be obtained by assuming ρ to depend on z, i.e., $\rho=\rho^{(t)}(z)$, and by requiring the functional stationarity condition

$$\frac{\delta F^{(t)}(\rho^{(t)},z)}{\delta\rho^{(t)}}=0 \tag{7.18}$$

(see Ref. [45] and references therein for a similar technique applied to the resummation of perturbative power expansions). The nontrivial fact, even surprising at first sight, is that the solution $\rho^{(t)}(z)$ of Eq. (7.18) is constant. In other words, for any t there exists a solution ρ_t independent of z that satisfies the global stationarity condition

$$\left.\frac{\partial F^{(t)}(\rho,z)}{\partial\rho}\right|_{\rho=\rho_t}=0. \tag{7.19}$$

This is equivalent to the fact that, for any universal ratio of amplitudes R, its approximation $R^{(t)}(\rho)$ [obtained from $F^{(t)}(\rho,z)$] satisfies the stationarity condition

$$\left.\frac{dR^{(t)}(\rho)}{d\rho}\right|_{\rho=\rho_t}=0. \tag{7.20}$$

The proof of Eq. (7.19) is given in Appendix C, where we show that the global stationarity condition amounts to requiring ρ_t to be a solution of the algebraic equation

$$\left[(2\beta-1)\rho\frac{\partial}{\partial\rho}-2\gamma+2t-2\right]h_{2t-1}(\rho)=0. \tag{7.21}$$

The idea behind our scheme of approximation is that, for any truncation, the stationarity condition enforces the physical request that the universal ratios of amplitudes be minimally dependent on ρ. To check the convergence of the approximation, one can repeat the computation of universal ratios of amplitudes from the truncated function $h^{(t)}(\rho_t,\theta)$ for different values of t, as long as one has a reliable estimate of F_{2t-1}. We have no *a priori* argument in favor of a fast convergence in t of the universal ratios of amplitudes derived by this procedure towards their exact values. However, we may appreciate that its lowest-order implementation, corresponding to $t=2$ in Eq. (7.15), reproduces the well known formulas of Refs. [24–26], which give an effective optimization of the linear parametric model. Indeed, we obtain from Eqs. (7.21) and (7.13) the $t=2$ solution

$$\rho_2=\sqrt{\frac{6\gamma(\gamma-1)}{\gamma-2\beta}}. \tag{7.22}$$

In this case the critical equation of state and all critical amplitudes turn out to be expressible simply in terms of the critical exponents β and γ. In particular, we found a closed form expression for all $F^{(2)}_{2m+1}$ coefficients (see Appendix C for a derivation):

$$F^{(2)}_{2m+1}=\frac{(-1)^m}{m!}\,\frac{\gamma(\gamma-1)}{\rho_2^{2m}}\prod_{k=1}^{m-2}(2\beta m-\gamma-k). \tag{7.23}$$

Wallace and Zia [27] already noticed that the minimum condition of Refs. [24–26] was equivalent to a condition of global stationarity for the linear parametric model. We have shown that such a global stationarity can be extended to other parametric models with $t\neq 2$, and can be used to improve the approximation.

The next truncation, corresponding to $t=3$, can also be treated analytically. Since it sensibly improves the linear parametric model in the 3D Ising case, we shall present here a few details. By applying the stationarity condition (7.21) to Eq. (7.14), we obtain

$$\rho_3=\sqrt{\frac{(\gamma-2\beta)(1-\gamma+2\beta)}{12(4\beta-\gamma)F_5}}\times\left(1-\sqrt{1-\frac{72(2-\gamma)\gamma(\gamma-1)(4\beta-\gamma)F_5}{(\gamma-2\beta)^2(1-\gamma+2\beta)^2}}\right)^{1/2}. \tag{7.24}$$

Universal ratios of amplitudes may be evaluated in terms of ρ_3; they will now depend only on the parameters β, γ, and F_5. Note that the predictions of the $t=2$ and $t=3$ models differ from each other only proportionally to the difference between the ‘‘experimental’’ value of F_5 and the value predicted according to Eq. (7.23),

$$F_5^{(2)}=\frac{(\gamma-2\beta)^2}{72\gamma(\gamma-1)}. \tag{7.25}$$

If we replace F_5 with $F_5^{(2)}$ in the $t=3$ model results, all the linear parametric model results are automatically reproduced. In the 3D Ising model, the two values differ by $\sim 6\%$, and thus we expect comparable discrepancies for all universal ratios of amplitudes. This can be verified from the numerical results that we will present in Sec. VII D (see Table XII). All universal ratios of amplitudes obtained from the $t=2$ truncation (i.e., the linear parametric model), using our estimates of γ and β, differ at most by a few per cent from previously available estimates. The $t=3$ and higher-order approximations are consistent with the latter. The apparent convergence in t of the results provides a further important support to this scheme.

It is worth noting that the parametric representation of the equation of state induces parametric forms for such thermodynamic functions as the free energy and the susceptibility, as discussed in detail in Appendix B 2. When we assume a truncated form of the parametric equation of state, in general only the corresponding free-energy function will admit a polynomial representation. A peculiar and possibly unique feature of our scheme is the induced truncation of the function related to the susceptibility, which turns out to be an even polynomial of degree $2t$ in the variable θ. Appendix C contains a more extended discussion of these and other properties of the approximation scheme based on the stationarity condition.

We have introduced our parametric representation assuming independent knowledge of $F_5,\ldots,F_{2t-1}$. It should be noted that our results for $t=3$ can also be used as a phenomenological parametrization, fitting the value of F_5 on any known universal quantity. As we will show in Sec. VII D, the difference with the linear parametric model of Refs. [24–

TABLE XII. Universal ratios of amplitudes obtained by taking different approximations of the parametric function $h(\theta)$. Numbers marked with an asterisk are inputs, not predictions. The values $\rho_{m,t}$ are obtained as in Ref. [23]; see text for details.

	$h^{(2)}(\rho_2,\theta)$	$h^{(3)}(\rho_3,\theta)$	$h^{(4)}(\rho_4,\theta)$	$h^{(5)}(\rho_5,\theta)$	$h^{(3)}(\rho_{m,3},\theta)$	$h^{(4)}(\rho_{m,4},\theta)$
ρ	1.7358(12)	1.7407(14)	1.7289(83)	1.686(51)	1.6889(26)	1.651(30)
θ_0^2	1.3606(11)	1.3879(29)	1.372(12)	1.325(53)	1.3310(13)	1.295(27)
F_0^∞	0.03280(14)	0.03382(18)	0.03374(21)	0.03366(26)	0.03378(18)	0.03370(23)
$\|z_0\|$	2.825(12)	2.7937(17)	2.7970(33)	2.8012(72)	2.7955(15)	2.7992(45)
U_0	0.5222(16)	0.5316(21)	0.5295(29)	0.5261(60)	0.5303(19)	0.5276(39)
U_2	4.826(11)	4.752(15)	4.769(22)	4.797(47)	4.764(13)	4.786(30)
U_4	−9.737(41)	−8.918(83)	−9.10(18)	−9.42(48)	−9.061(67)	−9.31(28)
R_c^+	0.05538(13)	0.05681(16)	0.05644(32)	0.0558(10)	0.05656(14)	0.05606(53)
R_c^-	0.021976(16)	0.022488(30)	0.02235(11)	0.02211(36)	0.022387(18)	0.02220(19)
R_4^+	7.9789(64)	7.804(10)	7.823(18)	7.847(40)	7.8146(85)	7.836(25)
R_4^-	92.10(23)	93.91(20)	93.25(45)	91.9(1.7)	93.25(21)	92.27(83)
v_3	6.0116(79)	6.0561(68)	6.041(11)	6.010(39)	6.0412(73)	6.018(20)
v_4	16.320(55)	16.121(66)	16.21(11)	16.41(24)	16.239(55)	16.38(15)
$Q_1^{-\delta}$	1.6775(19)	1.6588(27)	1.6624(44)	1.668(10)	1.6611(25)	1.6656(60)
$U_2R_4^+$	38.505(82)	37.09(15)	37.31(26)	37.64(55)	37.23(13)	37.50(35)
$R_4^+R_c^+$	0.4419(13)	0.4434(13)	0.4416(18)	0.4377(56)	0.4420(13)	0.4392(29)
r_6	1.9389(48)	*2.048(5)	*2.048(5)	*2.048(5)	*2.048(5)	*2.048(5)
r_8	2.507(31)	2.402(39)	*2.28(8)	*2.28(8)	2.365(43)	*2.28(8)
r_{10}	−12.612(41)	−12.146(60)	−10.0(1.5)	*−13(4)	−11.80(10)	−10.98(86)

26 is not negligible. On the other hand, our numerical estimates for $t=4$ show that the difference from $t=3$ is too small (compared with both theoretical and experimental precision) to justify the introduction of an additional phenomenological parameter F_7.

C. ϵ expansion of the parametric representation

It is interesting to compare our results with the analysis of the parametric equation of state, which can be performed in the context of the ϵ expansion, generalizing results presented in Refs. [27,112].

According to Ref. [27], within the ϵ expansion it is possible to choose a value ρ_0 such that for all $n\geqslant 2$,

$$h_{2n+1}(\rho_0)=O(\epsilon^{n+1}). \tag{7.26}$$

The calculation shows that $\rho_0=\sqrt{2}$. We proved in Appendix C that Eq. (7.26) keeps holding for all choices of ρ that satisfy the relation $\rho=\rho_0+O(\epsilon)$. We can now ϵ-expand our globally stationary solutions for arbitrary t, obtaining

$$\lim_{\epsilon\to 0}\rho_t=\rho_0. \tag{7.27}$$

As a consequence, any truncation satisfying the stationarity condition is an accurate description of the ϵ-expanded parametric equation of state up to and including $O(\epsilon^t)$.

As a byproduct, we may extract from the linear model relation (7.23), expanded to $O(\epsilon^2)$, the coefficients of the ϵ expansion for F_{2m+1}, for $m\geqslant 2$:

$$F_{2m+1}=\sum_{k=1}^{\infty} f_{mk}\epsilon^k. \tag{7.28}$$

We easily obtained from Eq. (7.23) the closed form results

$$f_{m1}=\frac{(-1)^m}{6m(m-1)}\frac{1}{2^m}, \tag{7.29}$$

$$f_{m2}=f_{m1}\left[\frac{17}{27}-\frac{m}{2}-\left(\frac{m}{3}+\frac{1}{6}\right)\sum_{k=1}^{m-2}\frac{1}{k}\right], \tag{7.30}$$

reproducing known results [23]. More generally, knowing the expansion of the coefficients F_{2m+1} to $O(\epsilon^t)$ for $m<t$ is enough to reconstruct all F_{2m+1} for $m\geqslant t$ with the same accuracy.

D. Results

As input parameters for the determination of the functions $h^{(t)}(\rho,\theta)$ we use the results of the IHT expansion: $\gamma=1.2371(4)$, $\nu=0.63002(23)$, $r_6=2.048(5)$, $r_8=2.28(8)$, $r_{10}=-13(4)$.

In Table XII we report the universal ratios of amplitude as derived from truncations corresponding to $t=2,3,4,5$. We use the standard notation for the ratios of amplitudes (see, e.g., Ref. [113]); all definitions can be found in Table XIII. For comparison, we also report, for $t=3,4$, the results obtained using the procedure of Ref. [23], fixing ρ to the value $\rho_{m,t}$ that minimizes the absolute value of the $O(\theta^{2t-1})$ coefficient. [As already noted in Ref. [23], for $t=4$ the minimum of $h_7(\rho)$ is zero, while for $t=3$, $h_5(\rho)$ never reaches zero.] Such results are very close to those derived from the stationarity condition; this is easily explained by the fact that the values of $\rho_{m,t}$ are close to ρ_t.

The errors reported in Table XII are related to the uncertainty of the corresponding input parameters (considering them as independent). The results for $t=2,3,4$ suggest a good convergence and give good support for our analysis. The results for $t=5$, although perfectly consistent, are less

TABLE XIII. Summary of the results obtained in this paper by our high-temperature calculations (IHT), by using the parametric representation of the equation of state (IHT-PR), by analyzing the low-temperature expansion (LT), and by combining the two approaches (IHT-PR+LT). Notations are explained in Appendix B.

	IHT	IHT-PR	LT	IHT-PR+LT
γ	1.2371(4)			
ν	0.63002(23)			
α	0.1099(7)			
η	0.0364(4)			
β	0.32648(18)			
δ	4.7893(22)			
σ	0.0208(12)			
r_6	2.048(5)			
r_8	2.28(8)			
r_{10}	−13(4)	−10(2)		
$U_0 \equiv A^+/A^-$		0.530(3)		
$U_2 \equiv C^+/C^-$		4.77(2)		
$U_4 \equiv C_4^+/C_4^-$		−9.1(2)		
$R_c^+ \equiv \alpha A^+ C^+/B^2$		0.0564(3)		
$R_c^- \equiv \alpha A^- C^-/B^2$		0.02235(11)		
$R_4^+ \equiv -C_4^+ B^2/(C^+)^3 = \lvert z_0 \rvert^2$		7.82(2)		
$R_3 \equiv v_3 \equiv -C_3^- B/(C^-)^2$		6.041(11)		
$R_4^- \equiv C_4^- B^2/(C^-)^3$		93.3(5)		
$v_4 \equiv -R_4^- + 3R_3^2$		16.21(11)		
$Q_1^{-\delta} \equiv R_\chi \equiv C^+ B^{\delta-1}/(\delta C^c)^\delta$		1.662(5)		
F_0^∞ cf. Eq. (7.4)		0.0337(2)		
$g_4^+ \equiv g \equiv -C_4^+/[(C^+)^2(f^+)^3]$	23.49(4)			
$w^2 \equiv C^-/[B^2(f^-)^3]$			4.75(4) [14]	
$U_\xi \equiv f^+/f^- = (w^2 U_2 R_4^+/g_4^+)^{1/3}$				1.961(7)
$Q^+ \equiv \alpha A^+(f^+)^3 = R_4^+ R_c^+/g_4^+$		0.01880(8)		
$R_\xi^+ \equiv (Q^+)^{1/3}$		0.2659(4)		
$Q^- \equiv \alpha A^-(f^-)^3 = R_c^-/w^2$				0.00471(5)
$Q_c \equiv B^2(f^+)^3/C^+ = Q^+/R_c^+ = R_4^+/g_4^+$		0.3330(10)		
$g_3^- \equiv w v_3$				13.17(6)
$g_4^- \equiv w^2 v_4$				77.0(8)
$Q_\xi^+ \equiv f_{\rm gap}^+/f^+$	1.000183(2)			
$Q_\xi^- \equiv f_{\rm gap}^-/f^-$			1.032(4) [39]	
$U_{\xi_{\rm gap}} \equiv f_{\rm gap}^+/f_{\rm gap}^- = U_\xi Q_\xi^+/Q_\xi^-$				1.901(10)
$Q_\xi^c \equiv f_{\rm gap}^c/f^c$				1.024(4)
$Q_2 \equiv (f^c/f^+)^{2-\eta} C^+/C^c$				1.195(10)

useful for checking convergence, due to the large uncertainty of F_9. In Table XIII we report our final estimates, obtained using $h^{(4)}(\rho_4,\theta)$; all the approximations reported in Table XII are consistent with them, except $t=2$.

We should say that the method of Guida and Zinn-Justin to determine the optimal ρ leads to equivalent results, and shows an apparent good convergence as well. However, we believe that the global stationarity represents a more physical requirement, and it is more amenable to a theoretical analysis of its convergence properties. Moreover, as we have shown, it has the linear parametric model of Refs. [24–27] as the lowest-order approximation.

Estimates of other universal ratios of amplitudes can be obtained by supplementing the above results with the estimates of $w^2 \equiv C^-/[B^2(f^-)^3]$ and $Q_\xi^- \equiv f_{\rm gap}^-/f^-$ obtained by an analysis of the corresponding low-temperature expansion. The results so obtained are denoted by IHT-PR+LT in Table XIII. The low-temperature expansion of w^2 can be calculated to $O(u^{21})$ on the cubic lattice using the series published in Refs. [109,114]. The results reported in Table XIII were obtained by using the Roskies transform in order to reduce the systematic effects due to confluent singularities [14].

We also consider a parametric representation of the correlation length. Following Ref. [115], we write

$$\xi^2/\chi = R^{-\eta\nu} a(\theta), \tag{7.31}$$

$$\xi_{\rm gap}^2/\chi = R^{-\eta\nu} a_{\rm gap}(\theta). \tag{7.32}$$

We consider the simplest polynomial approximation to $a(\theta)$ and $a_{\rm gap}(\theta)$:

$$a(\theta)\approx a_0(1+c\theta^2), \tag{7.33}$$

$$a_{\rm gap}(\theta)\approx a_{\rm gap,0}(1+c_{\rm gap}\theta^2), \tag{7.34}$$

where the constants c and $c_{\rm gap}$ can be determined by fitting the quantities U_ξ and $U_{\xi_{\rm gap}}$. Then, using Eqs. (7.31), (7.32), and the parametric representation of the equation of state, one can estimate the universal ratios of amplitudes Q_ξ^c and Q_2 defined in Table XIII. Notice that, given the equation of state, the normalization a_0 is not arbitrary, but it may be fixed using the zero-momentum four-point coupling g_4:

$$a_0=(h_0/\rho)^{1/3}(m_0/\rho)^{-5/3}(g_4^*)^{-2/3}, \tag{7.35}$$

where h_0, m_0 and ρ have been introduced in Eqs. (7.5) and (7.6). Notice that a_0 depends only on the ratios h_0/ρ and m_0/ρ, as is required of a physical quantity. Moreover, one has $a_{\rm gap,0}=(Q_\xi^+)^2 a_0$. In order to check the results obtained from the approximate expressions (7.33) and (7.34), we also considered the following parametric representation [102]:

$$\xi^{-2}=R^{2\nu}b(\theta), \tag{7.36}$$

$$\xi_{\rm gap}^{-2}=R^{2\nu}b_{\rm gap}(\theta), \tag{7.37}$$

and the corresponding polynomial approximations truncated to second order. The results for Q_ξ^c and Q_2 obtained by this second representation are perfectly consistent with those from the first one. Our final estimates of Q_ξ^c and Q_2 derived by the above method are reported in Table XIII.

In Table XIV we compare our results with other approaches. We find good overall agreement.

Our results appear to substantially improve the estimates of most of the universal ratios considered. In Table XIV we have collected results obtained by high-temperature and low-temperature expansions (HT,LT), Monte Carlo simulations (MC), field-theoretical methods such as ϵ expansion and various kinds of expansions at fixed dimension $d=3$, and experiments. Concerning the HT,LT estimates, we mention the recent Ref. [113], where a review of such results is presented. The agreement with the most recent Monte Carlo simulations is good, especially with the results reported in Ref. [122], which are quite precise. However, we note that the estimates of U_0 reported in Ref. [40] are slightly larger. Moreover, there is an apparent discrepancy with the estimate of g_4^- of Ref. [133]. It is worth mentioning that the result of Ref. [123] was obtained simulating a four-dimensional $SU(2)$ lattice gauge model at finite temperature, whose phase transition is expected to be in the 3D Ising universality class. Field-theoretical estimates are, in general, less precise, although perfectly consistent. We mention that the results denoted by ''$d=3$ exp.'' are obtained from different kinds of expansions: g expansion [23,44,121,132], minimal renormalization without ϵ expansion [120,135], expansion in the coupling $u\equiv 3w^2$ defined in the low-temperature phase [126]. In Refs. [23,44] Guida and Zinn-Justin used the $d=3$ g and ϵ expansions to calculate the small-field expansion of the effective potential and a parametric representation of the critical equation of state. We also mention the results (not included in this table) of an approach based on the approximate solution of exact renormalization equations (see, e.g., Refs. [48,49,136]). Some results can be found in Ref. [136]: $U_2\cong 4.29$, $g_3^-\cong 15.24$, $Q_1^{-\delta}\cong 1.61$, and $U_\xi \cong 1.86$.

We report in Table XIV experimental results for three interesting physical systems exhibiting a critical point belonging to the 3D Ising universality class: binary mixtures, liquid-vapor transitions and uniaxial antiferromagnetic systems. A review of experimental data can be found in Ref. [55]. Most of the results shown in Table XIV were reported in Refs. [23,122]. They should give an overview of the level of precision reached by experiments.

For the sake of comparison, in Table XV we report the universal ratios of amplitudes for the two-dimensional Ising model. The purely thermal results are taken from Ref. [105], where the exact two-point function has been written in terms of the solution of a Painlevé equation. Q^+ and Q^- have been computed by us solving numerically the differential equations reported in Ref. [105]. The ratios involving amplitudes along the critical isotherm can be obtained using the results reported in Ref. [138]. For the quantities that are not known exactly, we report estimates derived from the high- and low-temperature expansions. Such estimates are quite accurate and should be reliable because the leading correction to scaling is analytic, since the subleading exponent Δ is expected to be larger than one (see, e.g., Ref. [139] and references therein). In particular, the available exact calculations [105] for the square-lattice Ising model near criticality have shown only analytic corrections to the leading power law. Therefore, the traditional methods of series analysis should work well.

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APPENDIX A: GENERATION AND ANALYSIS OF THE HIGH-TEMPERATURE EXPANSION FOR IMPROVED HAMILTONIANS

1. Definitions

Before discussing the series computation, let us define all the quantities we are interested in and fix the notation.

Starting from the two-point function $G(x)\equiv\langle\phi(0)\phi(x)\rangle$, we define its spherical moments

$$m_{2j}=\sum_x (x^2)^j G(x) \tag{A1}$$

($\chi\equiv m_0$) and the first nonspherical moments

$$q_{4,2j}=\sum_x (x^2)^j[x^4-\tfrac{3}{5}(x^2)^2]G(x) \tag{A2}$$

(where $x^n\equiv\sum_i x_i^n$).

TABLE XIV. Estimates of the quantities in Table XIII by various approaches. The experimental data are taken from Ref. [55], unless otherwise stated. MS denotes a magnetic system; BM a binary mixture; LV the liquid-vapor transition in a simple fluid. For values marked with an asterisk, the error is not quoted explicitly in the reference.

	IHT–PR	HT,LT	MC	ϵ exp.	$d=3$ exp.	experiments	
U_0	0.530(3)	0.523(9) [116]	0.560(10) [40]	0.527(37) [44]	0.537(19) [44]	BM	0.56(2)
		0.51* [117]	0.550(12) [40]	0.524(10) [118,119]	0.540(11) [120]	LV	0.50(3)
			0.567(16) [40]		0.541(14) [121]	MS	0.51(3)
						LV [56]	0.53^{+8}_{-7}
						LV [57]	0.538(17)
U_2	4.77(2)	4.95(15) [116]	4.75(3) [122]	4.73(16) [44]	4.79(10) [44]	BM	4.4(4)
		5.01* [102]	4.72(11) [123]	4.9* [118]	4.77(30) [121]	LV	4.9(2)
				4.8* [124,125]	4.72(17) [126]	MS	5.1(6)
						MS [73]	4.6(2)
U_4	−9.1(2)	−9.0(3) [96]		−8.6(1.5) [44]	−9.1(6) [44]		
R_c^+	0.0564(3)	0.0581(10) [96]		0.0569(35) [44]	0.0574(20) [44]	BM	0.050(15)
					0.0594(10) [121]	LV	0.047(10)
R_4^+	7.82(2)	7.94(12) [113]		8.24(34) [23]	7.84* [44]		
$R_3 \equiv v_3$	6.041(11)	6.44(30) [113,96]		5.99(5) [86]	6.08(6) [44]		
				6.07(19) [44]			
R_4^-	93.3(5)	107(13) [96,113]					
v_4	16.21(11)			15.8(1.4) [86]			
$Q_1^{-\delta}$	1.662(5)	1.57(23) [127,113]		1.648(36) [44]	1.669(18) [44]	BM	1.75(30)
				1.67* [118,119]	1.7* [121]	LV	1.69(14)
w^2		4.75(4) [14]	4.77(3) [122]		4.73* [126]		
		4.71(5) [96,128]					
U_ξ	1.961(7)	1.96(1) [116]	1.95(2) [122]	1.91* [124]	2.013(28) [126]	BM	1.93(7)
		1.96* [102]	2.06(1) [129]			MS	1.92(15)
Q^+	0.01880(8)	0.0202(9) [130]	0.0193(10) [40]	0.0197* [131,119]	0.01968(15) [132]	LV [58]	0.0174(32)
		0.01880(15) [116]				LV [56]	0.023(4)
Q^-	0.00471(5)	0.00477(20) [113]	0.00463(17) [40]				
Q_c	0.3330(10)	0.324(6) [113]	0.328(5) [122]		0.331(9) [121]	BM	0.33(5)
						LV	0.35(4)
g_3^-	13.17(6)	13.9(4) [96]	13.6(5) [133]	13.06(12) [86]			
g_4^-	77.0(8)	85* [96]	108(7) [133]	75(7) [86]			
Q_ξ^+	1.000183(2)	1.0001* [113]		1.00016(2)	1.00021(3)		
Q_ξ^c	1.024(4)	1.007(3) [113]					
Q_ξ^-		1.032(4) [39]	1.031(6) [110,134]				
		1.037(3) [113]					
Q_2	1.195(10)	1.17(2) [127,113]		1.13* [124]			

The second-moment correlation length is defined by

$$\xi^2 \equiv M^{-2} = \frac{m_2}{6\chi}. \qquad \text{(A3)}$$

The coefficients c_i of the low-momentum expansion of the function $g(y)$ introduced in Sec. VI can be related to the critical limit of appropriate dimensionless ratios of spherical moments, or of the corresponding weighted moments $\overline{m}_{2j} \equiv m_{2j}/\chi$. Introducing the quantities

$$u_{2j} = \frac{1}{(2j+1)!}\overline{m}_{2j} M^{2j}, \qquad \text{(A4)}$$

TABLE XV. Universal ratios of amplitudes for the two-dimensional Ising model. Since the specific heat diverges logarithmically in the two-dimensional Ising model, the specific heat amplitudes $A^{\pm}$ are defined by $C_H \approx -A^{\pm} \ln t$.

γ	7/4
ν	1
$U_0 \equiv A^+/A^-$	1
$U_2 \equiv C^+/C^-$	37.69365201
$R_c^+ \equiv A^+C^+/B^2$	0.31856939
$R_c^- \equiv A^-C^-/B^2$	0.00845154
$Q_1^{-\delta} \equiv R_\chi \equiv C^+B^{\delta-1}/(\delta C^c)^\delta$	6.77828502
$w^2 \equiv C^-/[B^2(f^-)^2]$	0.53152607
$U_\xi \equiv f^+/f^-$	3.16249504
$U_{\xi_{\rm gap}} \equiv f^+_{\rm gap}/f^-_{\rm gap}$	2
$Q^+ \equiv A^+(f^+)^2$	0.15902704
$Q^- \equiv A^-(f^-)^2$	0.015900517
$Q_\xi^+ \equiv f^+_{\rm gap}/f^+$	1.000402074
$Q_\xi^c \equiv f^c_{\rm gap}/f^c$	1.0786828
$Q_\xi^- \equiv f^-_{\rm gap}/f^-$	1.581883299
$Q_2 \equiv (f^c/f^+)^{2-\eta}C^+/C^c$	2.8355305
$g_4^+ \equiv g \equiv -C_4^+/[(C^+)^2(f^+)^2]$	14.694(2) [137,14]
r_6	3.678(2) [99]
r_8	26.0(2) [99]
r_{10}	275(15) [99]
$v_3 \equiv -C_3^- B/(C^-)^2$	33.011(6) [96,86]
$v_4 \equiv -C_4^- B^2/(C^-)^3 + 3v_3^2$	48.6(1.2) [86]

one can define combinations of u_{2j} (that we will still call c_i to avoid introducing new symbols) having c_i as a critical limit:

$$c_2 = 1 - u_4 + \tfrac{1}{20}M^2, \tag{A5}$$

$$c_3 = 1 - 2u_4 + u_6 - \tfrac{1}{840}M^4, \tag{A6}$$

$$c_4 = 1 - 3u_4 + u_4^2 + 2u_6 - u_8 + \tfrac{1}{4725}M^4 + \tfrac{1}{60480}M^6, \tag{A7}$$

etc. Notice that the terms proportional to powers of M^2 do not contribute in the critical limit $M \to 0$, but they allow us to define improved estimators [39]. Indeed, in the lattice Gaussian limit, defined by the two-point function

$$\tilde{G}(k) = \frac{1}{\hat{k}^2 + M^2}, \quad \hat{k}^2 = \sum_i 4\sin^2(k_i/2), \tag{A8}$$

$c_i = 0$ independently of M, and not only in the critical limit $M \to 0$.

The zero-momentum connected Green's functions are defined by

$$\chi_{2j} = \sum_{x_2,\ldots,x_{2j}} \langle \phi(0)\phi(x_2)\ldots\phi(x_{2j-1})\phi(x_{2j})\rangle_c; \tag{A9}$$

in particular, $\chi_2 \equiv \chi$.

2. Linked cluster expansion

We computed the high-temperature expansion by the linked cluster expansion (LCE) technique. A general introduction to the LCE can be found in Ref. [140]. We modeled the application of the LCE to O(N)-symmetric models after Ref. [141].

In order to perform the LCE for the most general model described by the Hamiltonian (1.2), we parametrize the potential $V(\phi^2)$ in terms of the "single-site moments" [141]

$$\mathring{m}_{2k} = \frac{\Gamma(\frac{1}{2}N)}{2^k\Gamma(\frac{1}{2}N + k)} \frac{J_{N-1+2k}}{J_{N-1}},$$

$$J_k = \int_0^\infty dx\, x^k \exp[-V(x^2)]. \tag{A10}$$

We compute our series for fixed N, leaving all $\mathring{m}_{2k}$ as free parameters: each term of the series is a polynomial in $\mathring{m}_{2k}$ with rational coefficients.

With the aim of computing as many terms of the series as possible, we adopted all the technical developments of Ref. [142], and we introduced more improvements of our own; in this Appendix, we will only describe these; readers not familiar with technical details of the LCE should consult Refs. [141,142].

As discussed in Ref. [142], Sec. 3, the LCE requires a unique representation of graphs; it is convenient to implement this by defining a canonical form for the incidence matrix. The reduction to canonical form of a graph with V vertices requires, in principle, the comparison of the $V!$ incidence matrices obtained by permutation of vertices, which is clearly unmanageable for large graphs; even with the introduction of the "extended vertex ordering" of Ref. [142], this operation remains the dominant factor in the computation time; therefore, we devoted a large effort to the optimization of this aspect of the computation. On one hand, we have perfected the extended vertex ordering, and we are able to recognize inequivalent vertices much more often. On the other hand, we search for (a subgroup of) the symmetry group of the incidence matrix, which allows us to perform explicitly only one permutation for each equivalence class. Altogether, the largest sets of vertices which are explicitly permuted are of size 5 or less (except for a few hundred diagrams requiring 6, and a handful requiring permutations of 7 or 8 elements).

The next most computer-intensive operation is the computation of embedding numbers and color factors; it is optimized by "remembering" each computed value in a table, compatibly with available memory. This is crucial for color factors, which are computed recursively, and very effective for embedding numbers.

The problem of handling integer and rational quantities which do not fit into machine precision is solved by using the GNU multiprecision (GMP) library. Neither multiprecision nor polynomials in $\mathring{m}_{2k}$ are necessary for the most expensive sections of the computation; therefore, they have little impact on the computation time.

In order to speed up the handling of search and insertion into ordered sets of data, we make extensive use of AVL

TABLE XVI. Summary of normalization and length of our IHT series.

$\mathcal{O}$	r	n
χ	0	20
ξ^2	1	19
g_4	$-\frac{3}{2}$	17
r_6	0	17
r_8	0	16
r_{10}	0	15
c_2	4	13
c_3	3	13
c_4	2	13

trees (height-balanced binary trees) (cf., e.g., Ref. [143], Chap. 6.2.3). AVL trees are used to manipulate graph sets, multivariate polynomials, and tables of embedding numbers and of color factors.

The LCE is dramatically simplified by restricting actual computations to the set of one-particle irreducible graphs. One must, however, establish the relationship between the usual moments and susceptibilities and their irreducible parts. For this purpose, we found it convenient to define a generating functional of irreducible moments (irreducible momentum-space two-point functions):

$$G^{1\text{pi}}(\vec{p})=\sum_{\vec{x}} \exp(i\vec{p}\cdot\vec{x})G^{1\text{pi}}(\vec{x}), \tag{A11}$$

where $G^{1\text{pi}}(\vec{x})$ is the irreducible-graph contribution to the field-field correlation. One may then prove the relationship

$$[G(\vec{p})]^{-1}=[G^{1\text{pi}}(\vec{p})]^{-1}-2\beta\sum_{i=1}^{d} \cos p_i . \tag{A12}$$

By expanding both sides of Eq. (A12) in powers of p_i^2, it is trivial to establish all desired relationships, both for spherical and for nonspherical moments.

We have calculated χ and m_2 to 20th order, and the other moments of the two-point functions to 19th order. We have calculated χ_4 to 18th order, χ_6 to 17th order, χ_8 to 16th order, and χ_{10} to 15th order. Using Eqs. (3.12), (5.23), (5.24), and (5.25), one can obtain the HT series corresponding to the zero-momentum four-point coupling g_4 and the quantities r_{2j} that parametrize the effective potential.

It is useful to factorize out the leading dependence on β:

$$\mathcal{O}=\beta^r\sum_{i=0}^{n} a_i\beta^i; \tag{A13}$$

the values of r and n are summarized in Table XVI. In the following, we will analyze the series normalized to start with $O(\beta^0)$, i.e., $a_0+\beta a_1+\cdots$.

We have checked our series for χ_{2n} and m_2 against the available series of the standard Ising model (see, e.g., Refs. [13,36]); in this special case our only new result is the 18th-order coefficient of the expansion of χ_4:

$$a_{18}(\chi_4)=-\frac{171450770247965944104542584}{32564156625}. \tag{A14}$$

We have checked the (new) series for m_4 and m_6 by changing variables to $v=\tanh\beta$ and verifying that all coefficients become integer numbers.

It would be pointless to present here the full results for an arbitrary potential: the resulting expressions are only fit for further computer manipulation. For the three potentials we are interested in, we computed $\mathring{m}_{2k}$ by numerical integration (to 32-digit precision or higher). The coefficients a_i of the HT series for $\lambda_6=0$ and $\lambda_4=1.1$ and for $\lambda_6=1$ and $\lambda_4=1.9$ are reported in Tables XVII and XVIII, respectively. The series for the spin-1 model defined by Eq. (1.4), with $D=0.641$, are reported in Table XIX.

3. Critical exponents

In order to determine the critical exponent γ from the nth-order series of χ ($n=20$ in our case), we used quasidiagonal first-, second-, and third-order integral approximants (IA1's, IA2's, and IA3's, respectively).

IA1's are solutions of the first-order linear differential equation

$$P_1(x)f'(x)+P_0(x)f(x)+R(x)=0, \tag{A15}$$

where the functions $P_i(x)$ and $R(x)$ are polynomials that are determined by the known nth-order small-x expansion of $f(x)$. We considered $[m_1/m_0/k]$ IA1's with

$$m_1+m_0+k+2\geqslant n-p,$$

$$\max[\lfloor (n-p-2)/3\rfloor-q,2]\leqslant m_1,m_0,k\leqslant\lceil (n-p-2)/3\rceil+q, \tag{A16}$$

where m_1,m_0,k are the orders of the polynomial P_1, P_0, and R, respectively. The parameter q determines the degree of off-diagonality allowed. Since the best approximants are expected to be those that are diagonal or quasidiagonal, we considered sets of approximants corresponding to $q=3$. For a given integer number p, only approximants using $\overline{n}$ terms with $n\geqslant\overline{n}\geqslant n-p$ are selected by (A16). In our analysis we considered the values $p=0,1$.

IA2's are solutions of the second-order linear differential equation

$$P_2(x)f''(x)+P_1(x)f'(x)+P_0(x)f(x)+R(x)=0. \tag{A17}$$

We considered $[m_2/m_1/m_0/k]$ IA2's with

$$m_2+m_1+m_0+k+4\geqslant n-p,$$

$$\max[\lfloor (n-p-4)/4\rfloor-q,2]\leqslant m_2,m_1,m_0,k \leqslant\lceil (n-p-4)/4\rceil+q, \tag{A18}$$

where m_2,m_1,m_0,k are the orders of the polynomial P_2, P_1, P_0, and R, respectively. Again, the parameter q deter-

TABLE XVII. IHT series for $\lambda_4=1.1$, $\lambda_6=0$, in the notation of Eq. (A13).

i	χ	m_2	m_4	m_6
0	0.5308447611308816674	0	0	0
1	1.6907769625206168952	1.6907769625206168952	1.6907769625206168952	1.6907769625206168952
2	4.8619910172171602715	10.770481113538254496	28.721282969435345322	86.163848908306035967
3	13.927143445449825611	48.231864289335148202	219.75546850876176116	1168.8527451895890195
4	38.903779467013842036	188.05200645833592066	1229.8912709035843796	9662.7116928978662527
5	108.53608309082300208	675.87104272683156690	5832.6111092485591617	61434.000941321774177
6	299.26769419241406575	2309.0702869335329515	24922.163253442953617	332091.05589205489865
7	824.59140866646260666	7605.2927810004027352	99136.392920539417813	1607441.8553355166225
8	2256.9464691160956608	24394.253637363046531	374199.26312351789406	7181423.6410152125984
9	6174.4168460205032479	76627.209216394784414	1356978.6661855495156	30189095.008610024020
10	16819.879385593690953	236799.07667260657765	4767392.0099162457155	120979394.99942695905
11	45803.222727034040360	721928.56294418965891	16324368.332600723787	466443860.83571669282
12	124363.26835432977776	2176629.7412550662641	54721422.909210908059	1742031434.2158738860
13	337573.74963124787949	6500509.9999124014627	180180889.59217314660	6334414835.2189134577
14	914347.51881247417342	19258195.825678901432	584286774.86828839115	22515209536.395375901
15	2476042.0363235919004	56653368.820414865245	1869882856.0802601457	78474598792.492084546
16	6694111.3933182426254	165647959.63348214176	5915646880.4853995649	268883672555.38373998
17	18094604.163418613844	481713425.26839816320	18526280962.623843359	907568914837.87389317
18	48847832.893538572297	1394159442.3995129568	57500007423.420861038	3022858145406.1028395
19	131848611.02423050678	4017559436.5586218326	177034534120.13444060	9949496764882.0179674
20	355511932.47075480765	11532862706.754267638		

i	χ_4	χ_6	χ_8	χ_{10}
0	−0.3285640660980093563	1.0162413020868264428	−6.8394260743547676540	79.348906365011205720
1	−4.1859963164157358115	25.898050097125655128	−286.46285381596139097	4926.0498824266722100
2	−30.741724996686361292	334.99400966656728040	−5647.7911946128472205	136420.74020456605398
3	−176.53992137927974557	3085.9534765513322166	−75010.539622248383529	2448228.6617818973385
4	−873.19795113342604018	22962.676420970454675	−772240.19869798230118	33066114.972427672856
5	−3914.7539681033628945	147391.51074600633293	−6637987.3112014615524	364115447.75387624436
6	−16340.897140343714854	848040.97611583569625	−49819302.751247368336	3432806859.5766444770
7	−64653.470284596876205	4484314.0227663949556	−336218289.35412445864	28624021619.321695067
8	−245234.96688614286659	22168058.403461618364	−2082668486.5197392127	215985650524.12674877
9	−899257.84241359785957	103719954.32133484536	−12019618575.926765056	1499822923734.9247399
10	−3206654.1029660245146	463530204.01159412237	−65362330202.998798196	9707913378518.1866160
11	−11170819.137408319309	1992634695.4221763266	−337846725947.59204461	59157394345479.522579
12	−38148679.051940544866	8285182135.0991054145	−1671339685352.6240713	342085313031114.78303
13	−128071730.18983471414	33466763808.875455279	−7957523515361.0083551	1889271366312617.3703
14	−423602975.57444761466	131799942528.27775884	−36629909644439.962983	10018120044594804.748
15	−1382909952.8269485885	507558776082.59290672	−163635571877509.11631	51230076380872446.008
16	−4462746050.5347000940	1915992506452.6475673	−711670345253605.66031	
17	−14253923929.146690502	7104558940304.9779228		
18	−45107295178.923296542			

mines the degree of off-diagonality allowed, and we considered sets of approximants corresponding to $q=2$.

IA3's are solutions of the third-order linear differential equation

$$P_3(x)f'''(x)+P_2(x)f''(x)+P_1(x)f'(x)+P_0(x)f(x)+R(x)=0. \tag{A19}$$

We considered $[m_3/m_2/m_1/m_0/k]$ IA3's with

$$m_3+m_2+m_1+m_0+k+6\geqslant n-p,$$

$$\max[\lfloor(n-p-6)/5\rfloor-q,2]\leqslant m_3,m_2,m_1,m_0,k \leqslant\lceil(n-p-6)/5\rceil+q. \tag{A20}$$

We considered sets of approximants with $q=2$.

Our estimate of β_c and γ from each set of IA's is the average of the values corresponding to all nondefective approximants listed above. The error we quote is the standard deviation. Approximants are considered defective when they present spurious singularities close to the real axis for

TABLE XVIII. IHT series for $\lambda_4=1.9$, $\lambda_6=1$, in the notation of Eq. (A13).

i	χ	m_2	m_4	m_6
0	0.4655662671465330507	0	0	0
1	1.3005116946285418792	1.3005116946285418792	1.3005116946285418792	1.3005116946285418792
2	3.2867727127185864813	7.2656925005834656853	19.375180001555908494	58.125540004667725483
3	8.2759367806706747301	28.571904795057895903	130.05174486699400177	691.57352659837378755
4	20.312529070243777317	97.872800052605172528	638.95885613696143199	5016.1872942503349323
5	49.794483203882242272	309.06347996559814197	2661.2142140348739306	27994.182028926323346
6	120.62506581503040408	927.75225884375746610	9988.4118850888212117	132873.57808445335775
7	292.01512515747113432	2684.8185172507997345	34904.461272349558702	564842.72150668529468
8	702.16648194884216675	7566.3506306935978232	115746.99820735924226	2216511.1678444057633
9	1687.6428772292658916	20882.094002198352809	368765.50096064887178	8184889.2748500168622
10	4038.7968304000570954	56696.707453278695709	1138240.4288922759401	28813837.691500628877
11	9662.2729936573649464	151863.26982182684635	3424267.0781090006844	97595910.383064147131
12	23047.113364799224307	402271.59171959125975	10084793.611992545198	320214529.50558023009
13	54959.286082391555970	1055489.7357595885681	29174001.507844433518	1022942183.3309332937
14	130774.53710516575464	2747201.8540323841754	83116951.940376136421	3194364390.8578671172
15	311110.31236133784486	7100121.4902629495116	233696696.59215911285	9781470144.1262582176
16	738903.79800753751923	18238478.826399953589	649550962.77873855718	29444730149.930149105
17	1754633.3818777485475	46596282.414619468624	1787187905.0885303101	87315557903.862730209
18	4161219.5431591307916	118476705.93261076006	4873253901.0331542961	255504682223.76959562
19	9867152.2571694621736	299943150.21579862108	13181855628.291951390	738841191899.23449896
20	23372660.203375142541	756429264.37368967452		

i	χ_4	χ_6	χ_8	χ_{10}
0	−0.2477796481363361878	0.6474598784982264035	−3.7217906767025756640	37.013405312193356664
1	−2.7685883005851708908	14.535362477178409902	−137.07512921231932765	2014.8224919025960131
2	−17.877560582981223820	165.70820207079800267	−2380.8220701405889901	49093.219551382446508
3	−90.322805654272940178	1345.3663489182523069	−27871.670743361006737	776123.99513244411557
4	−393.02065426992610848	8820.7362245901247130	−252935.23058597306885	9238516.9347113237042
5	−1549.8790048885508754	49871.943864090224636	−1916250.8876237147160	89671059.994338257912
6	−5689.6695707018226942	252682.48564411688205	−12673128.761295226766	745143260.15631620580
7	−19795.290447377859541	1176295.0381597851008	−75349625.332312049631	5475799689.5566930641
8	−66017.086587684140210	5118124.4639866386541	−411110139.91029709454	36408447077.502609221
9	−212822.79826473961810	21072833.518843848267	−2089380048.3140434305	222743859156.05535943
10	−667124.90202267593913	82859426.257839273325	−10003697691.834647146	1270011540667.3854746
11	−2042817.2956666846385	313350700.53757124062	−45518090240.849158501	6816107886501.6215858
12	−6131769.6806946007770	1146009747.0203113140	−198194900838.32752342	34708877742973.050540
13	−18092493.782248051554	4071313753.6304055865	−830439274834.30354664	168778204360271.32357
14	−52592383.442563299219	14100289090.570869350	−3363665155462.8191575	787889916506321.20775
15	−150889630.61165148373	47747821664.976757633	−13220617132779.560155	3546559365689180.8887
16	−427911822.28715837917	158482857336.16489875	−50583038860319.049950	
17	−1201047651.7022285867	516675010346.30966717		
18	−3339910306.5273359851			

Re $\beta \lesssim \beta_c$. More precisely, we considered defective the approximants with spurious singularities in the rectangle

$$x_{\min} \leqslant \mathrm{Re}\, z \leqslant x_{\max}, \quad |\mathrm{Im}\, z| \leqslant y_{\max} \tag{A21}$$

where $z \equiv \beta/\beta_c$. The special values of $x_{\min}$, $x_{\max}$, and $y_{\max}$ are fixed essentially by stability criteria, and may differ in the various analysis. One should always check that, within a reasonable and rather wide range of values, the results depend very little on the values of $x_{\min}$, $x_{\max}$, and $y_{\max}$. The condition (A21) cannot be too strict, otherwise only a few approximants are left. In this case the analysis would be less robust and therefore less reliable. In the analysis of the critical exponents we fixed $x_{\min}=0.5$, $x_{\max}=1.5$, and $y_{\max}=0.5$. Sometimes we also eliminated seemingly good approximants whose results were very far from the average of the other approximants (more than three standard deviations).

As a further check of our analysis we used the fact that χ must present an antiferromagnetic singularity at $\beta_c^{\mathrm{af}} = -\beta_c$ with exponent $1-\alpha$ [37]; cf. Eq. (4.1). We verified the existence of a singularity at $\beta \simeq -\beta_c$ and calculated the associated exponent. In some analyses we selected the approximants with a pair of singularities β_c^{af} and β_c such that $\beta_c + \beta_c^{\mathrm{af}} \leqslant \varepsilon \beta_c$, and extracted the estimates of β_c, γ, and γ^{af}

TABLE XIX. IHT series for the spin-1 model at $D=0.641$, in the notation of Eq. (A13).

i	χ	m_2	m_4	m_6
0	0.5130338416658140921	0	0	0
1	1.5792223361663016211	1.5792223361663016211	1.5792223361663016211	1.5792223361663016211
2	4.4354864269385182067	9.7223340236143130564	25.926224062971501484	77.778672188914504451
3	12.419491590434972647	42.346809834993589030	191.98340105778667094	1019.9725391572417242
4	33.791261055416279831	160.98579902461367310	1043.4445460560890431	8166.3561104899782974
5	91.841711514904201711	564.00666776024395394	4815.3070263215122748	50394.717412597175181
6	246.45252550898370464	1877.9601316327756011	20036.716263907461491	264816.64944314665340
7	661.02891687705666766	6026.3998915743002481	77637.764412960183453	1247156.1972913768310
8	1760.2485042097784968	18829.052334178552758	285474.20654850401533	5423869.9597014869631
9	4685.9023162918237050	57600.716340522523798	1008432.9506590528288	22201293.939743630952
10	12417.403239568002677	173324.21633191405061	3450900.8811683331178	86641778.653884117489
11	32897.780017066145091	514450.09422679607285	11508712.418722055563	325332157.49685048174
12	86884.987650751268743	1509919.6022647962218	37570398.804037029475	1183310452.1936542300
13	229424.66023796502223	4389267.7826683406013	120463565.05045829114	4190401181.7453112798
14	604434.19630362500628	12656102.176997207352	380359458.94391796446	14504931859.074136829
15	1592166.9939650416411	36234119.703394609364	1185138499.8785854689	49231114687.307644210
16	4186778.2089201191352	103100174.19850927716	3650157580.5925683557	164257098550.90866386
17	11008100.036706697979	291755921.33251277078	11128154630.539032580	539841179613.24402666
18	28904025.427069749972	821639224.91959656471	33620312241.342947618	1750686524137.2717781
19	75884596.302083003892	2303832207.2589619187	100754983048.94595013	5610143692354.6457755
20	199011100.35574405792	6434727599.1288912159		

i	χ_4	χ_6	χ_8	χ_{10}
0	−0.2765773264173367185	0.6159505110893571460	−2.9991913236448423261	25.639540414004216349
1	−3.4054446789491065718	15.965825500576724169	−131.09722874868945636	1673.3057480542363748
2	−24.570159964990933901	209.17712832988124972	−2673.1181642081994199	48487.693538927944393
3	−139.04496175368819963	1941.9314061871553985	−36418.744278635469309	903571.05878070985095
4	−676.69722203050291907	14481.161465713991310	−381670.70958440882592	12572456.840428388939
5	−2980.2850555040833295	92716.594093707793992	−3318236.6956045245656	141625856.39478899275
6	−12201.170328998898251	530071.30088368454939	−25054434.002489226202	1357653370.1441467390
7	−47288.798001661947371	2776591.5196504503986	−169364792.32951428214	11451941832.665412740
8	−175521.23292975181789	13563137.626324266820	−1047051748.2193837515	87035901992.179110029
9	−629296.76861756838811	62579486.626554750976	−6012877960.8045829709	606512902870.52199480
10	−2192523.1579334804197	275333754.56012219662	−32454249850.587780052	3927249840355.5223369
11	−7458569.3308517637566	1163638939.7290930525	−166147643973.94469398	23876098793430.398990
12	−24861185.382016374834	4751112394.9532757501	−812614050679.29384404	137426012654323.52351
13	−81432521.735317351753	18826990715.719661290	−3819167751639.5573175	753928052790611.90171
14	−262699929.49158922336	72675832066.845002904	−17330670355620.376973	3964188356224407.7875
15	−836235938.80657455801	274128146369.26786810	−76232334051550.698537	20070120847424617.843
16	−2630659142.3082426455	1012932475761.7342902	−326121067148253.92416	
17	−8189047799.9075180970	3674558299661.5473376		
18	−25252383565.446882139			

from them. As in Ref. [8], we also considered IA's where the polynomial associated with the highest derivative of $f(x)$ is even, i.e., it is a polynomial in x^2. We will denote them by $b_{\rm af}$ IA's. This ensures that if x_c is a singularity of an approximant, then $-x_c$ is also a singular point.

In Table XX we present the results for some values of the parameters p,q,ε introduced above (when the value of ε is not explicitly shown it means that the corresponding constraint was not implemented). We quote the ''ratio of approximants'' $\mathcal{R}_{\rm app}$ $(l-s)/t$, where t is the total number of approximants in the given set, l is the number of nondefective approximants [passing the test (A21)], and s is the number of seemingly good approximants which are excluded because their results are very far from the other approximants; $l-s$ is the number of ''good'' approximants used in the analysis; note that $s \ll l$, and $l-s$ is never too small. We found the IA2 analysis to give the most stable results, especially with respect to the change of the number of terms of the series considered. Therefore, we consider the IA2 results to be the most reliable. Moreover, IA1's are not completely satisfactory in reproducing the antiferromagnetic singularity when its presence is not biased. In the biased analyses where

TABLE XX. Results of various analyses of the 20th-order IHT series for χ. $\mathcal{R}_{\rm app}$ is explained in the text. In the biased analyses forcing the value of β_c, the error is reported as a sum: the first term is related to the spread of the approximants at the central value of β_c, while the second one is related to the uncertainty of the value of β_c and it is estimated by varying β_c.

	Approx.	$\mathcal{R}_{\rm app}$	β_c	γ	$\beta_c^{\rm af}$	$\gamma^{\rm af}$
$\lambda_6=0,\ \lambda_4=1.08$	$\text{IA2}_{q=2,p=0}$	(78−2)/85	0.3760701(22)	1.23717(36)	−0.376(6)	−1.0(7)
	$\text{IA3}_{q=2,p=0}$	(62−1)/65	0.3760699(21)	1.23713(37)	−0.377(4)	−1.0(3)
	$b_{\rm af}\text{IA3}_{q=2,p=0}$	(63−1)/65	0.3760703(18)	1.23719(30)		−0.903(10)
$\lambda_6=0,\ \lambda_4=1.10$	$\text{IA1}_{q=3,p=0}$	30/37	0.3750945(27)	1.23684(42)	−0.375(3)	−0.9(3)
	$\text{IA1}_{q=3,p=0,\varepsilon=10^{-2}}$	(26−1)/37	0.3750955(18)	1.23699(26)	−0.3760(8)	−0.77(8)
	$\text{IA1}_{q=3,p=0,\varepsilon=10^{-3}}$	1/37	0.3750956	1.23701	−0.3754	−0.81
	$b_{\rm af}\text{IA1}_{q=3,p=0}$	(24−2)/37	0.3750937(46)	1.23673(69)		−0.887(11)
	$\text{IA2}_{q=2,p=0}$	(78−2)/85	0.3750975(18)	1.23734(29)	−0.376(6)	−1.0(6)
	$\text{IA2}_{q=2,p=0,\varepsilon=10^{-2}}$	(69−1)/85	0.3750974(21)	1.23733(35)	−0.375(2)	−0.9(2)
	$\text{IA2}_{q=2,p=0,\varepsilon=10^{-3}}$	16/85	0.3750975(19)	1.23735(31)	−0.3751(2)	−0.90(4)
	$b_{\rm af}\text{IA2}_{q=2,p=0}$	(54−1)/85	0.3750989(31)	1.23753(46)		−0.906(18)
	$\text{IA2}_{q=2,p=1}$	(151−3)/165	0.3750974(26)	1.23731(39)	−0.376(4)	−1.0(5)
	$\text{IA2}_{q=2,p=1,\varepsilon=10^{-3}}$	25/165	0.3750975(16)	1.23735(25)	−0.3751(2)	−0.90(3)
	$\text{IA3}_{q=2,p=0}$	(62−1)/65	0.3750971(21)	1.23728(36)	−0.375(4)	−1.0(5)
	$\text{IA3}_{q=2,p=0,\varepsilon=10^{-3}}$	21/65	0.3750983(12)	1.23749(19)	−0.3750(2)	−0.92(2)
	$b_{\rm af}\text{IA3}_{q=2,p=0}$	(63−1)/65	0.3750976(18)	1.23734(30)		−0.902(11)
	$\text{IA3}_{q=2,p=1}$	(99−3)/100	0.3750957(55)	1.23704(98)	−0.375(3)	−1.0(4)
	$b_{\beta_c^{\rm MC}}\text{IA1}_{q=3,p=0}$	38/48	0.3750966(4) [20]	1.23718(7+7)		
	$b_{\beta_c^{\rm MC}}\text{IA2}_{q=2,p=0}$	(114−3)/115	0.3750966(4) [20]	1.23720(2+7)		
	$b_{\pm\beta_c^{\rm MC}}\text{IA2}_{q=2,p=0}$	(90−2)/100	0.3750966(4) [20]	1.23719(3+7)		
	$b_{\pm\beta_c^{\rm MC}}\text{IA3}_{q=2,p=0}$	(61−1)/63	0.3750966(4) [20]	1.23719(8+7)		
$\lambda_6=0,\ \lambda_4=1.12$	$\text{IA2}_{q=2,p=0}$	(77−2)/85	0.3741203(16)	1.23748(26)	−0.374(4)	−1.0(6)
	$\text{IA3}_{q=2,p=0}$	(62−1)/65	0.3741199(21)	1.23742(36)	−0.375(4)	−1.0(3)
	$b_{\rm af}\text{IA3}_{q=2,p=0}$	(62−1)/65	0.3741203(17)	1.23747(29)		−0.904(9)
$\lambda_6=1,\ \lambda_4=1.86$	$\text{IA2}_{q=2,p=0}$	(82−3)/85	0.4307605(22)	1.23676(30)	−0.431(5)	−1.1(5)
$\lambda_6=1,\ \lambda_4=1.90$	$\text{IA1}_{q=3,p=0}$	37/37	0.4269723(81)	1.2363(11)	−0.429(7)	−0.7(5)
	$b_{\rm af}\text{IA1}_{q=3,p=0}$	(29−1)/37	0.426972(7)	1.2363(9)		−0.890(16)
	$\text{IA2}_{q=2,p=0}$	(83−2)/85	0.4269779(24)	1.23711(34)	−0.428(5)	−1.0(5)
	$\text{IA2}_{q=2,p=0,\varepsilon=10^{-3}}$	23/85	0.4269779(32)	1.23710(45)	−0.4271(2)	−0.90(3)
	$b_{\rm af}\text{IA2}_{q=2,p=0}$	(69−2)/85	0.4269791(32)	123725(48)		−0.903(16)
	$\text{IA2}_{q=2,p=1}$	(156−4)/165	0.4269777(32)	1.23707(42)	−0.428(5)	−0.9(5)
	$\text{IA2}_{q=2,p=1,\varepsilon=10^{-3}}$	31/165	0.4269777(30)	1.23708(41)	−0.4271(2)	−0.89(4)
	$\text{IA3}_{q=2,p=0}$	(63−1)/65	0.4269782(24)	1.23714(38)	−0.429(6)	−1.0(4)
	$b_{\rm af}\text{IA3}_{q=2,p=0}$	(63−1)/65	0.4269786(25)	1.23719(36)		−0.906(10)
$\lambda_6=1,\ \lambda_4=1.94$	$\text{IA2}_{q=2,p=0}$	(82−3)/85	0.4232606(21)	1.23738(30)	−0.423(6)	−1.1(9)
spin-1, $D=0.633$	$\text{IA2}_{q=2,p=0}$	(82−4)/85	0.3845065(27)	1.23683(34)	−0.384(8)	−1.1(1.2)
	$b_{\rm af}\text{IA2}_{q=2,p=0}$	(73−4)/85	0.3845076(17)	1.23698(25)		−0.905(12)
spin-1, $D=0.641$	$b_{\rm af}\text{IA1}_{q=3,p=0}$	(30−3)/37	0.3856634(41)	1.2360(6)		−0.887(11)
	$\text{IA2}_{q=2,p=0}$	(73−2)/85	0.3856681(33)	1.23674(38)	−0.386(3)	−0.9(4)
	$\text{IA2}_{q=2,p=0,\varepsilon=10^{-3}}$	(37−2)/85	0.3856685(17)	1.23678(23)	−0.3860(3)	−0.87(3)
	$b_{\rm af}\text{IA2}_{q=2,p=0}$	(73−4)/85	0.3856691(16)	1.23687(25)		−0.905(12)
	$\text{IA2}_{q=2,p=1}$	(146−3)/165	0.3856669(53)	1.23655(69)	−0.385(10)	−1.2(2.0)
	$\text{IA3}_{q=2,p=0}$	(62−5)/65	0.3856686(36)	1.23678(58)	−0.385(4)	−1.1(5)
	$b_{\rm af}\text{IA3}_{q=2,p=0}$	(61−2)/65	0.3856682(38)	1.23673(60)		−0.911(13)
spin-1, $D=0.649$	$\text{IA2}_{q=2,p=0}$	(82−3)/85	0.3868365(32)	1.23660(36)	−0.386(7)	−1.2(1.0)
	$b_{\rm af}\text{IA2}_{q=2,p=0}$	(73−4)/85	0.3868377(16)	1.23676(24)		−0.905(12)

β_c is forced, IA1's, IA2's, and IA3's give almost equivalent results.

The results are quite stable, and the value of $\gamma^{\rm af}$ is always consistent with $1-\alpha\simeq 0.89$. From the results of Table XX, combining the results of the IA2 and IA3 analyses (selecting the results denoted by $\text{IA2}_{q=2,p=0}$, $b_{\rm af}\text{IA2}_{q=2,p=0}$, $\text{IA3}_{q=2,p=0}$, and $b_{\rm af}\text{IA3}_{q=2,p=0}$) we obtain the following estimates:

TABLE XXI. Results of the analysis of the 19th-order IHT series for ξ^2/β. $\mathcal{R}_{\rm app}$ is explained in the text. The error is reported as a sum of two terms. The first term is related to the spread of the approximants at the central value of β_c, while the second one is related to the uncertainty of the value of β_c and it is estimated by varying β_c.

	Approx.	$\mathcal{R}_{\rm app}$	ν	$\gamma^{\rm af}$
$\lambda_6=0,\ \lambda_4=1.08$	$b_{\beta_c}{\rm IA2}_{q=2,p=0}$	$(54-2)/70$	0.63004(2+11)	
$\lambda_6=0,\ \lambda_4=1.10$	$b_{\beta_c}{\rm IA1}_{q=3,p=0}$	35/37	0.63012(2+10)	
	$b_{\beta_c}{\rm IA2}_{q=2,p=0}$	51/70	0.63016(3+9)	
	$b_{\pm\beta_c}{\rm IA2}_{q=2,p=0}$	$(53-3)/55$	0.63015(5+10)	$-0.88(5)$
	$b_{\beta_c}{\rm IA3}_{q=2,p=0}$	28/35	0.63009(14+10)	
	$b_{\beta_c}{\rm IA2}_{q=2,p=1}$	$(86-4)/132$	0.63016(3+9)	
	$b_{\beta_c^{\rm MC}}{\rm IA2}_{q=2,p=0}$	$(47-2)/55$	0.63012(3+3)	
$\lambda_6=0,\ \lambda_4=1.12$	$b_{\beta_c}{\rm IA2}_{q=2,p=0}$	$(49-2)/70$	0.63027(3+9)	
$\lambda_6=1,\ \lambda_4=1.86$	$b_{\beta_c}{\rm IA2}_{q=1,p=0}$	$(60-1)/70$	0.62978(1+11)	
$\lambda_6=1,\ \lambda_4=1.90$	$b_{\beta_c}{\rm IA1}_{q=3,p=0}$	34/37	0.63000(2+12)	
	$b_{\beta_c}{\rm IA2}_{q=2,p=0}$	$(61-2)/70$	0.63003(2+11)	
	$b_{\pm\beta_c}{\rm IA2}_{q=2,p=0}$	$(55-2)/55$	0.63003(7+11)	$-0.87(16)$
	$b_{\beta_c}{\rm IA2}_{q=2,p=1}$	$(113-4)/132$	0.63003(3+10)	
	$b_{\beta_c}{\rm IA3}_{q=2,p=0}$	$(27-1)/34$	0.62988(17+14)	
$\lambda_6=1,\ \lambda_4=1.94$	$b_{\beta_c}{\rm IA2}_{q=2,p=0}$	$(55-2)/70$	0.63023(2+11)	
spin-1, $D=0.633$	$b_{\beta_c}{\rm IA2}_{q=2,p=0}$	$(67-1)/70$	0.62998(2+13)	
spin-1, $D=0.641$	$b_{\beta_c}{\rm IA1}_{q=3,p=0}$	33/37	0.62988(5+15)	
	$b_{\beta_c}{\rm IA2}_{q=2,p=0}$	$(66-1)/70$	0.62990(2+13)	
	$b_{\beta_c}{\rm IA3}_{q=2,p=0}$	24/35	0.62981(12+19)	
spin-1, $D=0.649$	$b_{\beta_c}{\rm IA2}_{q=2,p=0}$	$(66-1)/70$	0.62982(2+13)	

$$\beta_c=0.3750973(14),\quad \gamma=1.23732(24)\quad \text{for}\ \lambda_6=0,\quad \lambda_4=1.10, \tag{A22}$$

$$\beta_c=0.4269780(18),\quad \gamma=1.23712(26)\quad \text{for}\ \lambda_6=1,\quad \lambda_4=1.90, \tag{A23}$$

and

$$\beta_c=0.3856688(20),\quad \gamma=1.23680(30)\quad \text{for spin-1},\quad D=0.641. \tag{A24}$$

Also also into account the uncertainty of λ_4^* and D^*, we arrive at the estimates of Table III. Notice that the value of β_c at $\lambda_4=1.10$ and $\lambda_6=0$ is in agreement with the Monte Carlo estimate of Ref. [20], i.e., $\beta_c=0.3750966(4)$ (where according to the author the error does not include possible systematic errors). From the analysis of the antiferromagnetic singularity using the $b_{\rm af}$IA's we obtain the following estimate for α:

$$\alpha=0.105(10), \tag{A25}$$

which is in good agreement with the much more precise estimate (4.5) obtained assuming hyperscaling.

In order to determine ν from the analysis of the HT series of ξ^2, we followed the suggestion of Ref. [38], i.e., to use the estimate of β_c derived from the analysis of χ in order to bias the analysis of the series of ξ^2. We analyzed the 19th-order series of ξ^2/β and employed biased integral approximants (b_{β_c}IA). For instance, biased IA2's can be obtained from the solutions of the equation

$$(1-x/\beta_c)P_2(x)f''(x)+P_1(x)f'(x)+P_0(x)f(x)+R(x)=0. \tag{A26}$$

In this case we considered the approximants satisfying the conditions

$$\begin{aligned} &m_2+m_1+m_0+k+3\geqslant n-p,\\ &\max[\lfloor(n-p-3)/4\rfloor-q,2]\leqslant m_2,m_1,m_0,k\\ &\qquad\leqslant\lfloor(n-p-3)/4\rfloor+q, \end{aligned} \tag{A27}$$

where, as before, m_i and k are the orders of the polynomials P_i and R, respectively. We also tried doubly biased IA2 ($b_{\pm\beta_c}$IA2) where also a singularity at $-\beta_c$ is forced using solutions of the equation

$$(1-x^2/\beta_c^2)P_2(x)f''(x)+P_1(x)f'(x)+P_0(x)f(x)+R(x)=0. \tag{A28}$$

In Table XXI we report the results of some of the analyses we performed. In the case of the $b_{\pm\beta_c}$IA2 analysis we also report the exponent at the antiferromagnetic singularity which turned out to be always consistent with $1-\alpha$. The error of ν is given as a sum of two terms: the first one is computed from the spread of the approximants at β_c, the second one is related to the uncertainty of β_c.

TABLE XXII. Results for η obtained using the CPRM: (a) applied to ξ^2 and χ (20 orders); (b) applied to ξ^2/β and χ (19 orders). $\mathcal{R}_{\rm app}$ is explained in the text.

	Approx.	$\mathcal{R}_{\rm app}$	$\eta\nu$
$\lambda_6=0,\ \lambda_4=1.08$	(a) $b\mathrm{IA2}_{q=2,p=0}$	(95−1)/115	0.02274(13)
	(b) $b\mathrm{IA2}_{q=2,p=0}$	38/70	0.02294(8)
$\lambda_6=0,\ \lambda_4=1.10$	(a) $b\mathrm{IA2}_{q=2,p=0}$	95/115	0.02280(14)
	(a) $b\mathrm{IA2}_{q=2,p=1}$	(150−1)/185	0.02280(16)
	(a) $b\mathrm{IA3}_{q=2,p=0}$	(47−1)/61	0.02280(37)
	(b) $b\mathrm{IA1}_{q=3,p=0}$	28/37	0.02300(8)
	(b) $b\mathrm{IA2}_{q=2,p=0}$	36/70	0.02301(8)
	(b) $b\mathrm{IA2}_{q=2,p=1}$	86/132	0.02309(12)
	(b) $b\mathrm{IA3}_{q=2,p=0}$	31/34	0.02311(22)
$\lambda_6=0,\ \lambda_4=1.12$	(a) $b\mathrm{IA2}_{q=2,p=0}$	97/115	0.02285(15)
	(b) $b\mathrm{IA2}_{q=2,p=0}$	35/70	0.02308(9)
$\lambda_6=1,\ \lambda_4=1.86$	(a) $b\mathrm{IA2}_{q=2,p=0}$	(94−5)/115	0.02267(12)
	(b) $b\mathrm{IA2}_{q=2,p=0}$	39/70	0.02285(12)
$\lambda_6=1,\ \lambda_4=1.90$	(a) $b\mathrm{IA2}_{q=2,p=0}$	(90−2)/115	0.02278(12)
	(b) $b\mathrm{IA2}_{q=2,p=0}$	37/70	0.02298(11)
$\lambda_6=1,\ \lambda_4=1.94$	(a) $b\mathrm{IA2}_{q=2,p=0}$	(92−2)/115	0.02288(13)
	(b) $b\mathrm{IA2}_{q=2,p=0}$	32/70	0.02312(10)
spin-1, $D=0.633$	(a) $b\mathrm{IA2}_{q=2,p=0}$	(84−1)/115	0.02292(40)
	(b) $b\mathrm{IA2}_{q=2,p=0}$	36/70	0.02316(22)
spin-1, $D=0.641$	(a) $b\mathrm{IA2}_{q=2,p=0}$	(85−2)/115	0.02288(40)
	(b) $b\mathrm{IA2}_{q=2,p=0}$	37/70	0.02312(22)
spin-1, $D=0.649$	(a) $b\mathrm{IA2}_{q=2,p=0}$	(84−1)/115	0.02285(43)
	(b) $b\mathrm{IA2}_{q=2,p=0}$	37/70	0.02307(22)

We quote as our final estimates:

$$\nu=0.63015(12)\quad \text{for}\quad \lambda_6=0,\quad \lambda_4=1.10, \tag{A29}$$

$$\nu=0.63003(13)\quad \text{for}\quad \lambda_6=1,\quad \lambda_4=1.90, \tag{A30}$$

and

$$\nu=0.62990(15)\quad \text{for spin-1},\quad D=0.641. \tag{A31}$$

Also taking into account the uncertainty of λ_4^* and D^*, we arrive at the estimates of Table III. We mention that unbiased IA analyses of the 19th series of ξ^2/β give consistent but less precise estimates of β_c [cf. Eqs. (A22) and (A23)] and ν.

As a check of our results, we performed a biased analysis of χ and ξ^2 at $\lambda_6=0$ and $\lambda_4=1.10$, using the value $\beta_c=0.3750966(4)$ obtained in Ref. [20] by Monte Carlo simulations based on finite-size scaling techniques. Although the author of Ref. [20] says that the error on β_c does not include systematic errors, we used it as a check and found (see Tables XX and XXI) $\gamma=1.23720(2+7)$ and $\nu=0.63012(3+3)$ (the first error is related to the spread of the approximants at $\beta_c=0.3750966$ and the second one to the error on β_c), which are perfectly consistent with our final estimates reported in Table III.

In order to obtain an estimate of η without using the scaling relation $\gamma=(2-\eta)\nu$, we employed the so-called critical-point renormalization method (CPRM). The idea of the CPRM is that from two series $D(x)$ and $E(x)$ which are singular at the same point x_0, $D(x)=\Sigma_i d_i x^i \sim (x_0-x)^{-\delta}$ and $E(x)=\Sigma_i e_i x^i \sim (x_0-x)^{-\epsilon}$, one constructs a new series $F(x)=\Sigma_i (d_i/e_i) x^i$. The function $F(x)$ is singular at $x=1$ and for $x\to 1$ behaves as $F(x)\sim(1-x)^{-\phi}$, where $\phi=1+\delta-\epsilon$. Therefore, the analysis of $F(x)$ provides an unbiased estimate of the difference between the critical exponents of the two functions $D(x)$ and $E(x)$. The series $F(x)$ may be analyzed by employing biased approximants with a singularity at $x_c=1$. In order to check for possible systematic errors, we applied the CPRM to the series of ξ^2/β and χ (analyzing the corresponding 19th-order series) and to the series of ξ^2 and χ (analyzing the corresponding 20th-order series). We used IA's biased at $x_c=1$. In Table XXII we present the results of the analysis for some values of the parameters q,p. We obtain $\eta\nu=0.02294(20)$ at $\lambda_6=0$ and $\lambda_4=1.1$, $\eta\nu=0.02287(20)$ at $\lambda_6=1$ and $\lambda_4=1.9$, and $\eta\nu=0.02305(20)$ for spin-1 and $D=0.641$. Taking again into account the uncertainty of λ_4^* and D^* we then obtain the estimate reported in Table III.

The CPRM was also employed in order to estimate the exponent σ. It was applied to the 18th-order series of $\chi\xi^2/\beta$ and $q_{4,0}/\beta$. The results are displayed in Table XXIII. We find $\sigma\nu=0.0134(8)$ for $\lambda_6=0$ and $\lambda_4=1.1$, $\sigma\nu=0.0134(9)$ for $\lambda_6=1$ and $\lambda_4=1.9$, and $\sigma\nu=0.0127(6)$ for spin-1 at $D=0.641$.

4. Ratios of amplitudes

In the following we describe the analysis we employed in order to evaluate universal ratios of amplitudes, such as g_4, r_{2j}, and c_i, from the corresponding HT series. In the case of

TABLE XXIII. Results for σ obtained using the CPRM applied to m_2/β and $q_{4,0}/\beta$ (18 orders). Here we used $x_{\rm min}=0.9$, $x_{\rm max}=1.1$, $x_{\rm max}=0.1$. $\mathcal{R}_{\rm app}$ is explained in the text.

	Approx.	$\mathcal{R}_{\rm app}$	$\sigma\nu$
$\lambda_6=0$, $\lambda_4=1.08$	bIA2$_{q=1,p=0}$	29/34	0.0134(8)
$\lambda_6=0$, $\lambda_4=1.10$	bIA2$_{q=1,p=0}$	29/34	0.0134(8)
	bIA2$_{q=2,p=0}$	53/62	0.0133(10)
$\lambda_6=0$, $\lambda_4=1.12$	bIA2$_{q=1,p=0}$	28/34	0.0135(9)
$\lambda_6=1$, $\lambda_4=1.86$	bIA2$_{q=1,p=0}$	29/34	0.0133(9)
$\lambda_6=1$, $\lambda_4=1.90$	bIA2$_{q=1,p=0}$	29/34	0.0134(9)
	bIA2$_{q=2,p=0}$	52/62	0.0132(12)
$\lambda_6=1$, $\lambda_4=1.94$	bIA2$_{q=1,p=0}$	28/34	0.0135(9)
spin-1, $D=0.633$	bIA2$_{q=1,p=0}$	21/34	0.0127(5)
spin-1, $D=0.641$	bIA2$_{q=1,p=0}$	21/34	0.0127(5)
	bIA2$_{q=2,p=0}$	37/62	0.0128(6)
spin-1, $D=0.649$	bIA2$_{q=1,p=0}$	21/34	0.0126(5)

g, c_2, c_3, and c_4 we analyzed the series $\beta^{3/2}g_4=\Sigma_{i=0}^{17}a_i\beta^i$, $\beta^{-4}c_2=\Sigma_{i=0}^{13}a_i\beta^i$, $\beta^{-3}c_3=\Sigma_{i=0}^{13}a_i\beta^i$, and $\beta^{-2}c_4=\Sigma_{i=0}^{13}a_i\beta^i$. In order to obtain estimates of the universal critical limit of g_4, r_{2j}, and c_i, we evaluated the approximants of the corresponding HT series at β_c (as determined from the analysis of the magnetic susceptibility), and multiplied by the appropriate power of β_c.

For an nth-order series we considered three sets of approximants: Padé approximants (PA's), Dlog-Padé approximants (DPA's), and first-order integral approximants (IA1's).

(1) $[l/m]$ PA's with

$$l+m\geqslant n-2, \tag{A32}$$

$$\max[n/2-q,4]\leqslant l,\quad m\leqslant n/2+q, \tag{A33}$$

where l,m are the orders of the polynomials, respectively, in the numerator and denominator of the PA. The parameter q determines the degree of off-diagonality allowed. The best approximants should be those that are diagonal or quasidiagonal. So we considered PA's selected using $q=3$. Our final estimate from the PA's is the average of the values at β_c of the nondefective approximants using all the available terms of the series and satisfying the condition (A33) with $q=2$. The error we quote is the standard deviation of the results from all the nondefective approximants listed above. We considered defective PA's with spurious singularities in the rectangle defined in Eq. (A21) with $x_{\rm min}=0.9$ ($x_{\rm min}=0$ only in the case of r_{10}), $x_{\rm max}=1.01$, and $y_{\rm max}=0.1$.

(2) $[l/m]$ DPA's with

$$l+m\geqslant n-2,\qquad \max[(n-1)/2-q,4]\leqslant l,\quad m\leqslant(n-1)/2+q, \tag{A34}$$

where l,m are the orders of the polynomials, respectively, in the numerator and denominator of the PA of the series of its logarithmic derivative. We again fixed $q=3$. The estimate with the corresponding error is obtained as in the case of PA's. We considered defective DPA's with spurious singularities in the rectangle with $x_{\rm min}=0$, $x_{\rm max}=1.01$ and $y_{\rm max}=0.1$; cf. Eq. (A21).

(3) $[m_1/m_0/k]$ IA1's given by Eq. (A16). The off-diagonality parameter was fixed to be $q=3$, and $p=1$. Our final estimate is the average of the values at β_c of all nondefective approximants listed above. The error we quote is the standard deviation of the results. We considered defective IA1's with spurious singularities in the rectangle and $x_{\rm min}=0$, $x_{\rm max}=1.001$, and $y_{\rm max}=0.1$.

As in the case of the critical exponents, sometimes we also eliminated seemingly good approximants whose results were very far from the average of the other approximants. In order to arrive at a final estimate, the results from PA's, DPA's, and IA's were then combined, also taking into account the relative number of nondefective approximants (before combining the results we divided the apparent error of each set of approximants by the square root of the ratio between the number of nondefective and the total number of approximants). Of course, all of the above procedure used to arrive at a final estimate is rather subjective. But we believe it provides reasonable estimates of the quantity at hand and of its uncertainty. We report in Table XXIV the results of each set of approximants so that the readers can judge the reliability of our final estimates. The second error in the combined estimate is related to the uncertainty of the value of λ_4^* and D^*; it is estimated by varying λ_4 in the range 1.08–1.12 for $\lambda_6=0$, 1.86–1.94 for $\lambda_6=1$, and D in the range 0.633–0.649 for the spin-1 model. Errors due to the uncertainty of β_c are negligible.

We have also performed analyses of the series of g_4 for $\lambda_6=0$ and several values of λ_4 by employing the Roskies transform [11]. The idea of the Roskies transform (RT) is to perform biased analyses that take into account the leading confluent singularity. For the Ising model, where $\Delta\simeq1/2$, one replaces the variable β in the original expansion with a new variable z, defined by $1-z=(1-\beta/\beta_c)^{1/2}$. If the original series has square-root scaling correction terms, the transformed series has analytic correction terms, which can be handled by standard PA's or DPA's. Note that in principle IA1's should be able to detect the first nonanalytic correction

TABLE XXIV. Results of PA, DPA, and IA1 analyses of the series for g_4, r_{2j}, and c_i. When results are not reported, it means that for that quantity no acceptable results were obtained from that class of approximants. The fraction subscript is the number of nondefective approximants over the total number of approximants. The last column contains the estimates obtained by combining the three classes of approximants.

		PA	DPA	IA1	Combined
g_4^*	$\lambda_6=0,\ \lambda_4=1.10$	$23.500(60)_{17/21}$	$23.491(25)_{16/18}$	$23.504(18)_{49/73}$	23.499(16+20)
	$\lambda_6=1,\ \lambda_4=1.90$	$23.487(45)_{17/21}$	$23.474(46)_{17/18}$	$23.498(24)_{57/73}$	23.491(21+40)
	spin-1, $D=0.641$	$23.486(19)_{20/21}$	$23.492(88)_{17/18}$	$23.491(52)_{53/73}$	23.487(18+20)
r_6	$\lambda_6=0,\ \lambda_4=1.10$	$2.051(13)_{19/21}$	$2.058(12)_{11/18}$	$2.048(7)_{32/73}$	2.051(7+2)
	$\lambda_6=1,\ \lambda_4=1.90$	$2.052(12)_{18/21}$	$2.063(14)_{11/18}$	$2.048(4)_{33/73}$	2.050(5+4)
	spin-1, $D=0.641$	$2.0493(65)_{20/21}$	$2.0461(24)_{16/18}$	$2.0456(16)_{23/73}$	2.046(2+3)
r_8	$\lambda_6=0,\ \lambda_4=1.10$	$2.24(9)_{17/18}$	$2.21(13)_{17/21}$	$2.23(5)_{37/69}$	2.23(5+4)
	$\lambda_6=1,\ \lambda_4=1.90$	$2.23(11)_{18/18}$	$2.23(9)_{17/21}$	$2.23(5)_{36/69}$	2.23(5+6)
	spin-1, $D=0.641$	$2.40(8)_{16/18}$	$2.31(5)_{17/21}$	$2.42(13)_{26/69}$	2.34(5+3)
r_{10}	$\lambda_6=0,\ \lambda_4=1.10$	$-14(5)_{15/21}$		$-13.3(1.3)_{6/61}$	−14(4+0)
	$\lambda_6=1,\ \lambda_4=1.90$	$-14(5)_{14/21}$		$-12(4)_{10/61}$	−13(5+0)
	spin-1, $D=0.641$	$-10(21)_{13/21}$		$4(36)_{14/61}$	−8(25+0)
10^4c_2	$\lambda_6=0,\ \lambda_4=1.10$	$-3.582(8)_{15/15}$	$-3.580(29)_{12/12}$	$-3.586(24)_{24/33}$	−3.582(7+6)
	$\lambda_6=1,\ \lambda_4=1.90$	$-3.574(7)_{14/15}$	$-3.574(26)_{12/12}$	$-3.585(38)_{24/33}$	−3.574(7+20)
	spin-1, $D=0.641$	$-3.570(12)_{15/15}$	$-3.562(36)_{11/12}$	$-3.554(28)_{25/33}$	−3.568(11+4)
10^4c_3	$\lambda_6=0,\ \lambda_4=1.10$	$0.087(8)_{12/15}$	$0.080(10)_{3/12}$	$0.084(7)_{26/36}$	0.085(6+0)
	$\lambda_6=1,\ \lambda_4=1.90$	$0.086(5)_{11/15}$	$0.078(12)_{2/12}$	$0.086(5)_{26/36}$	0.086(4+0)
	spin-1, $D=0.641$	$0.095(14)_{14/15}$	$0.100(12)_{2/12}$	$0.090(4)_{30/36}$	0.090(4+0)

TABLE XXV. Details of the analysis of the 17th-order series for $\beta^{-3/2}g_4(\beta)$ with and without the use of the RT for some values of λ_4 and $\lambda_6=0$. In the PA, DPA, and IA analyses with RT we used $q=2$ (other approximants turned out to be much less stable). We fixed $x_{\min}=0$, $x_{\max}=1.1$, and $y_{\max}=0.25$ for PA and DPA, and $x_{\min}=0$, $x_{\max}=1.01$, and $y_{\max}=0.25$ for IA. In order to perform a homogeneous comparison, we used the same procedure for the direct analysis without RT (except that we used $x_{\max}=1.01$ and $y_{\max}=0.1$). The fraction subscript is the number of nondefective approximants over the total number of approximants.

λ_4		PA	DPA	IA	Combined
0.5	direct	$22.62(58)_{10/15}$	$22.43(16)_{9/12}$	$22.55(19)_{19/37}$	22.48(15)
	RT	$23.75(27)_{8/15}$	$23.48(25)_{8/12}$	$23.29(50)_{15/37}$	23.56(23)
0.7	direct	$23.04(28)_{10/15}$	$22.92(14)_{9/12}$	$22.95(13)_{15/37}$	22.94(12)
	RT	$23.62(31)_{13/15}$	$23.54(26)_{9/12}$	$23.35(35)_{17/37}$	23.54(20)
1.0	direct	$23.58(23)_{9/15}$	$23.40(6)_{8/12}$	$23.38(19)_{15/37}$	23.41(7)
	RT	$23.56(27)_{13/15}$	$23.57(15)_{9/12}$	$23.45(21)_{14/37}$	23.55(14)
1.1	direct	$23.500(64)_{12/15}$	$23.491(23)_{11/12}$	$23.494(16)_{17/37}$	23.493(16)
	RT	$23.56(22)_{13/15}$	$23.59(16)_{9/12}$	$23.44(17)_{14/37}$	23.55(13)
1.2	direct	$23.63(4)_{10/15}$	$23.613(9)_{11/12}$	$23.612(9)_{20/37}$	23.613(8)
	RT	$23.56(20)_{13/15}$	$23.54(34)_{11/12}$	$23.43(16)_{12/37}$	23.52(15)
1.5	direct	$23.93(4)_{14/15}$	$23.92(6)_{7/12}$		23.93(4)
	RT	$23.55(27)_{12/15}$	$23.53(31)_{11/12}$	$23.41(13)_{11/37}$	23.48(16)
2.0	direct	$24.14(28)_{15/15}$	$24.07(17)_{3/12}$	$24.15(9)_{31/37}$	24.15(9)
	RT	$23.61(22)_{12/15}$	$23.52(18)_{10/12}$	$23.44(12)_{17/37}$	23.50(12)
3.0	direct	$24.42(14)_{15/15}$	$24.14(39)_{6/12}$	$24.40(19)_{15/37}$	24.40(12)
	RT	$23.61(23)_{14/15}$	$23.47(14)_{11/12}$	$23.34(18)_{16/37}$	23.48(11)
∞	direct	$24.78(10)_{15/15}$	$24.57(19)_{10/12}$	$24.81(16)_{9/37}$	24.75(9)
	RT	$23.59(20)_{13/15}$	$23.47(11)_{10/12}$	$23.48(16)_{13/37}$	23.50(10)

to scaling, but they probably need many more terms of the series, and practically need to be explicitly biased as in the case of PA's and DPA's. Indeed the IA1 results without the RT turn out to be substantially equivalent to those obtained using PA's and DPA's. In Table XXV we report the details of the analysis without and with the RT for some values of λ_4 and $\lambda_6=0$. These results are plotted in Fig. 1.

APPENDIX B: UNIVERSAL RATIOS OF AMPLITUDES

1. Notations

Universal ratios of amplitudes characterize the behavior in the critical domain of thermodynamical quantities that do not depend on the normalizations of the external (e.g., magnetic) field, order parameter (e.g., magnetization), and temperature. Amplitude ratios of zero-momentum quantities can be derived from the critical equation of state. We consider several amplitudes derived from the singular behavior of the specific heat,

$$C_H=A^{\pm}|t|^{-\alpha}, \tag{B1}$$

the magnetic susceptibility,

$$\chi=C^{\pm}|t|^{-\gamma}, \tag{B2}$$

the spontaneous magnetization on the coexistence curve,

$$M=B|t|^{-\beta}, \tag{B3}$$

the zero-momentum connected n-point correlation functions,

$$\chi_n=C_n^{\pm}|t|^{-\gamma-(n-2)\beta\delta}. \tag{B4}$$

We complete our list of amplitudes by considering the second-moment correlation length

$$\xi=f^{\pm}|t|^{-\nu}, \tag{B5}$$

and the true (on-shell) correlation length, describing the large-distance behavior of the two-point function,

$$\xi_{\rm gap}=f^{\pm}_{\rm gap}|t|^{-\nu}. \tag{B6}$$

One can also define amplitudes along the critical isotherm, e.g.,

$$\chi=C^c|H|^{-\gamma/\beta\delta}, \tag{B7}$$

$$\xi=f^c|H|^{-\nu/\beta\delta}, \tag{B8}$$

$$\xi_{\rm gap}=f^c_{\rm gap}|H|^{-\nu/\beta\delta}. \tag{B9}$$

2. Universal ratios of amplitudes from the parametric representation

In the following we report the expressions of the universal ratios of amplitudes in terms of the parametric representation (7.5) of the critical equation of state.

The singular part of the free energy per unit volume can be written as

$$\mathcal{F}=h_0 m_0 R^{2-\alpha}g(\theta), \tag{B10}$$

where $g(\theta)$ is the solution of the first-order differential equation

$$(1-\theta^2)g'(\theta)+2(2-\alpha)\theta g(\theta)=(1-\theta^2+2\beta\theta^2)h(\theta) \tag{B11}$$

that is regular at $\theta=1$. One may also write

$$\chi^{-1}=\frac{h_0}{m_0}R^{\gamma}g_2(\theta),\quad g_2(\theta)=\frac{2\beta\delta\theta h(\theta)+(1-\theta^2)h'(\theta)}{(1-\theta^2+2\beta\theta^2)}, \tag{B12}$$

$$\chi_3=\frac{m_0}{h_0^2}R^{-2\gamma-\beta}g_3(\theta),$$

$$g_3(\theta)=-\frac{(1-\theta^2)g_2'(\theta)+2\gamma\theta g_2(\theta)}{g_2(\theta)^3(1-\theta^2+2\beta\theta^2)}, \tag{B13}$$

$$\chi_4=\frac{m_0}{h_0^3}R^{-3\gamma-2\beta}g_4(\theta),$$

$$g_4(\theta)=\frac{(1-\theta^2)g_3'(\theta)-2(2\gamma+\beta)\theta g_3(\theta)}{g_2(\theta)(1-\theta^2+2\beta\theta^2)}. \tag{B14}$$

Using the above formulas one can then calculate the universal ratios of amplitudes:

$$U_0\equiv\frac{A^+}{A^-}=(\theta_0^2-1)^{2-\alpha}\frac{g(0)}{g(\theta_0)}, \tag{B15}$$

$$U_2\equiv\frac{C^+}{C^-}=(\theta_0^2-1)^{-\gamma}\frac{g_2(0)}{g_2(\theta_0)}, \tag{B16}$$

$$u_4\equiv\frac{C_4^+}{C_4^-}=(\theta_0^2-1)^{-3\gamma-2\beta}\frac{g_4(0)}{g_4(\theta_0)}, \tag{B17}$$

$$R_c^+\equiv\frac{\alpha A^+C^+}{B^2}=-\alpha(1-\alpha)(2-\alpha)(\theta_0^2-1)^{2\beta}\theta_0^{-2}g(0), \tag{B18}$$

$$R_c^-\equiv\frac{\alpha A^-C^-}{B^2}=\frac{R_c^+}{U_0U_2}, \tag{B19}$$

$$v_3\equiv R_3\equiv-\frac{C_3^-B}{(C^-)^2}=-\theta_0 g_2(\theta_0)^2 g_3(\theta_0), \tag{B20}$$

$$v_4\equiv-\frac{C_4^-B^2}{(C^-)^3}+3v_3^2=\theta_0^2 g_2(\theta_0)^3[3g_2(\theta_0)g_3(\theta_0)^2 -g_4(\theta_0)], \tag{B21}$$

$$R_4^+\equiv-\frac{C_4^+B^2}{(C^+)^3}=|z_0|^2, \tag{B22}$$

$$Q_1^{-\delta}\equiv R_\chi\equiv\frac{C^+B^{\delta-1}}{(\delta C^c)^{\delta}}=(\theta_0^2-1)^{-\gamma}\theta_0^{\delta-1}h(1), \tag{B23}$$

$$F_0^\infty \equiv \lim_{z\to\infty} z^{-\delta} F(z) = \rho^{1-\delta} h(1). \tag{B24}$$

Using Eq. (7.7) one can compute $F(z)$ and obtain the small-z expansion coefficients of the effective potential r_{2j} in terms of the critical exponents and the coefficients h_{2l+1} of the expansion of $h(\theta)$.

APPENDIX C: APPROXIMATION SCHEME FOR THE PARAMETRIC REPRESENTATION OF THE EQUATION OF STATE BASED ON STATIONARITY

The parametric form of the critical equation of state, described by Eqs. (7.5), (7.6), and (7.7), shows a formal dependence on the auxiliary parameter ρ.

However all physically relevant amplitude ratios are independent of ρ, because they may be expressed in terms of the invariant function $F(z)$ and its derivatives, evaluated at such special values of z as $z=0$, $z=\infty$ and $z=z_0$, where $F(z_0)=0$. Notice that, despite the apparent dependence generated by the relation $z_0 \equiv z(\rho, \theta_0(\rho))$, from the definition it follows that z_0 must necessarily be independent of ρ.

We can exploit these facts to set up an approximation procedure in which the function $h(\rho,\theta)$, entering the scaling equation of state, is truncated to some simpler (polynomial) function $h^{(t)}(\rho,\theta)$ and the value of ρ is properly fixed to optimize the approximation.

We found that, at any given order in the truncation, it is possible and convenient to choose ρ in such a way that all the (truncated) universal amplitude ratios are simultaneously stationary against infinitesimal variations of ρ itself.

Starting from $h^{(t)}(\rho,\theta)$ we may reconstruct the function

$$\tilde{F}^{(t)}(\rho,\theta) = \frac{\rho h^{(t)}(\rho,\theta)}{(1-\theta^2)^{\beta+\gamma}}. \tag{C1}$$

In order that all truncated amplitudes be simultaneously stationary in ρ, it is necessary that the function $F^{(t)}(\rho,z) \equiv \tilde{F}^{(t)}(\rho,\theta(\rho,z))$ be stationary with respect to variations of ρ for any value of z.

We shall prove that for any polynomial truncation $h^{(t)}(\rho,\theta)$ it is possible to find a value ρ_t, independent of z, such that

$$\left.\frac{\partial F^{(t)}(\rho,z)}{\partial \rho}\right|_{\rho=\rho_t} = 0, \tag{C2}$$

a property which we shall term ‘‘global stationarity.’’

In order to prove our statement, let us rephrase the above condition into the form

$$\frac{\partial \tilde{F}^{(t)}(\rho,\theta)}{\partial \rho} + \frac{\partial \tilde{F}^{(t)}(\rho,\theta)}{\partial \theta}\frac{\partial \theta}{\partial \rho} = 0, \tag{C3}$$

where the implicit function theorem allows us to write

$$\frac{\partial \theta}{\partial \rho} = -\frac{\partial z/\partial \rho}{\partial z/\partial \theta}. \tag{C4}$$

The definitions (7.6) and (7.7) imply

$$\frac{\partial z}{\partial \rho} = \frac{z}{\rho}, \quad \frac{\partial z}{\partial \theta} = z\left(\frac{1}{\theta} + \frac{2\beta\theta}{1-\theta^2}\right). \tag{C5}$$

Moreover, it is trivial to show that

$$\frac{\partial \tilde{F}^{(t)}(\rho,\theta)}{\partial \rho} = \tilde{F}^{(t)}(\rho,\theta)\left(\frac{1}{\rho} + \frac{1}{h^{(t)}(\rho,\theta)}\frac{\partial h^{(t)}(\rho,\theta)}{\partial \rho}\right), \tag{C6}$$

$$\frac{\partial \tilde{F}^{(t)}(\rho,\theta)}{\partial \theta} = \tilde{F}^{(t)}(\rho,\theta)\left(\frac{2(\beta+\gamma)\theta}{1-\theta^2} + \frac{1}{h^{(t)}(\rho,\theta)}\frac{\partial h^{(t)}(\rho,\theta)}{\partial \theta}\right). \tag{C7}$$

Substitution of these expressions into Eq. (C3) leads to the following form of the global stationarity condition:

$$[1-(1+2\gamma)\theta^2]h^{(t)}(\rho,\theta) + [1+(2\beta-1)\theta^2]\rho\frac{\partial h^{(t)}(\rho,\theta)}{\partial \rho} - [1-\theta^2]\theta\frac{\partial h^{(t)}(\rho,\theta)}{\partial \theta} = 0. \tag{C8}$$

Let us now write down $h^{(t)}(\rho,\theta)$ as a power series in the odd powers of θ:

$$h^{(t)}(\rho,\theta) = \theta + \sum_{n=1}^{t-1} h_{2n+1}(\rho)\,\theta^{2n+1}. \tag{C9}$$

The series-expanded stationarity condition then takes the form

$$\sum_{n=1}^{t-1}\left\{\left[\rho\frac{\partial}{\partial\rho} - 2n\right]h_{2n+1}(\rho) + \left[(2\beta-1)\rho\frac{\partial}{\partial\rho} - 2\gamma + 2n - 2\right]h_{2n-1}(\rho)\right\}\theta^{2n+1} = 0, \tag{C10}$$

with the convention $h_1 = 1$.

Let us now note that in the absence of truncations the above equation must be identically true, since the original function $F(z)$ is totally independent of ρ. This fact implies that the coefficients of the above power-series expansion must vanish individually, and this gives us an infinite set of recursive differential equations for the functions $h_{2n+1}(\rho)$, which must be automatically satisfied when the coefficients $h_{2n+1}(\rho)$ are properly defined.

A truncation corresponds to arbitrarily suppressing all coefficients starting from $h_{2t+1}(\rho)$. Hence, the global stationarity condition simply amounts to requiring

$$\left\{\left[(2\beta-1)\rho\frac{\partial}{\partial\rho} - 2\gamma + 2t - 2\right]h_{2t-1}(\rho)\right\}\theta^{2t+1} = 0, \tag{C11}$$

because all other terms vanish. The resulting equation can be solved by choosing ρ_t such that the term in curly brackets vanishes, independent of θ. This concludes our proof.

The effectiveness of this scheme is beautifully illustrated by its lowest-order implementation, corresponding to the so-called ‘‘linear parametric model,’’ in the context of the three-dimensional Ising model.

Let us truncate the exact scaling function $h(\rho,\theta)$ to its cubic approximation

$$h^{(2)}(\rho,\theta)=\theta+h_3(\rho)\theta^3, \tag{C12}$$

where $h_3(\rho)$ is taken from Eq. (7.13). Substituting h_3 into the stationarity condition (C11) for $t=2$, we find

$$\frac{1}{3}(2\beta-\gamma)\rho^2-2\gamma(1-\gamma)=0, \tag{C13}$$

which leads to

$$\rho_2=\sqrt{\frac{6\gamma(\gamma-1)}{\gamma-2\beta}}. \tag{C14}$$

The truncated scaling function vanishes at the value θ_0:

$$\theta_0^2\equiv-\frac{1}{h_3(\rho_2)}=\frac{\gamma-2\beta}{\gamma(1-2\beta)}. \tag{C15}$$

In this approximation the scaling equation of state turns out to be expressible simply in terms of the critical exponents β and γ. As a consequence, all the universal ratios may then be approximated to lowest order by appropriate algebraic combinations of the critical exponents.

The above results reproduce the old formulas by Schofield, Lister, and Ho [25], who obtained expressions for critical amplitudes in terms of critical exponents from a minimum condition imposed on the predictions extracted from a parametric scaling equation of state.

In the case of the linear parametric model, the global nature of the stationarity property introduced by the above authors was shown by Wallace and Zia [27], who adopted a slightly different, but essentially equivalent, formulation of the above model.

As we showed above, global stationarity can be imposed on parametric models regardless of the linearity constraint. The next truncation, corresponding to $t=3$, can also be treated analytically. Our starting point will be

$$h^{(3)}(\rho,\theta)=\theta+h_3(\rho)\theta^3+h_5(\rho)\theta^5. \tag{C16}$$

The coefficients $h^{(3)}(\rho)$ and $h^{(5)}(\rho)$ are reported in Eqs. (7.13) and (7.14). By applying the stationarity condition (C11) to $h_5(\rho)$, we obtain

$$\rho_3=\sqrt{\frac{(\gamma-2\beta)(1-\gamma+2\beta)}{12(4\beta-\gamma)F_5}}\times\left(1-\sqrt{1-\frac{72(2-\gamma)\gamma(\gamma-1)(4\beta-\gamma)F_5}{(\gamma-2\beta)^2(1-\gamma+2\beta)^2}}\right)^{1/2}. \tag{C17}$$

The truncated scaling function vanishes when θ takes the value θ_0, which is now given by the relation

$$\theta_0^2=\frac{h_3(\rho_3)}{2h_5(\rho_3)}\left(\sqrt{1-\frac{4h_5(\rho_3)}{h_3^2(\rho_3)}}-1\right). \tag{C18}$$

A general feature of truncated parametric models is the possibility of making a prediction about higher-order coefficients F_{2n+1}, for $n\geqslant t$, in terms of lower-order coefficients. This is a natural consequence of having included by the parametrization some information on the asymptotic behavior of $F(z)$ for large z. In practice, we observe that each F_{2n+1} appears first in the coefficient h_{2n+1}, in the form of a free constant of integration in the solution of the recursive differential equation relating h_{2n+1} to h_{2n-1}. Since a truncation corresponds to setting $h_{2n+1}=0$ starting from $n=t$, this fixes the (truncated) value of all F_{2n+1} starting from F_{2t+1}.

As an important consequence of this mechanism, we observe that truncated models deviate from the exact solution only in proportion to the difference between exact and predicted coefficients, and this difference may be quite small even for very low-order truncations.

In order to turn the above considerations into quantitative estimates, we need to gain further insight into the properties of the functions $h_{2n+1}(\rho)$, especially in the vicinity of the stationary point ρ_t. To this end we introduce the expansion

$$h_{2n+1}(\rho)=\sum_{m=0}^{n}c_{n,m}\rho^{2m}F_{2m+1}. \tag{C19}$$

Substituting this expansion as an ansatz into the recursive differential equations, we check that F_{2m+1} act as free parameters (integration constants), while the coefficients $c_{n,m}$ must obey the following algebraic recursive equations:

$$(n-m)c_{n,m}=[(2\beta-1)m-\gamma+n-1]c_{n-1,m} \tag{C20}$$

for all $n>m$, subject to the initial conditions $c_{m,m}=1$. It is possible to find a closed-form solution to Eq. (C20),

$$c_{n,m}=\frac{1}{(n-m)!}\prod_{k=1}^{n-m}(2\beta m-\gamma+k-1), \tag{C21}$$

but for our purposes the recursive equations will sometimes be more useful than their explicit solutions.

Let us define the coefficients of the z expansion of the truncated scaling function evaluated at the stationary point:

$$F^{(t)}(\rho_t,z)=z+\tfrac{1}{6}z^3+\sum_{m=2}F_{2m+1}^{(t)}z^{2m+1}. \tag{C22}$$

By definition, $F_{2m+1}^{(t)}$ coincides with its exact value F_{2m+1} for all $m<t$, while for $m\geqslant t$ it is determined by the condition $h_{2m+1}(\rho_t)=0$, which, according to Eq. (C19), implies

$$\sum_{m=0}^{n}c_{n,m}\rho_t^{2m}F_{2m+1}^{(t)}=0 \tag{C23}$$

for all $n\geqslant t$.

We can now prove the following lemma:

$$\sum_{m=1}^{n}mc_{n,m}\rho_t^{2m}F_{2m+1}^{(t)}=0 \tag{C24}$$

holds for all $n\geqslant t$.

The proof is by induction. Let us assume the lemma to hold for a given value n; then, as a consequence of Eqs. (C24) and (C23), we obtain

$$\sum_{m=0}^{n} [(2\beta-1)m-\gamma+n]c_{n,m}\rho_t^{2m}F_{2m+1}^{(t)}=0. \quad \text{(C25)}$$

Notice that the above equation also holds for the initial value $n=t-1$, since in that case it coincides with the global stationarity condition.

By use of the recursion equations (C20) we now obtain

$$\sum_{m=0}^{n} (n+1-m)c_{n+1,m}\rho_t^{2m}F_{2m+1}^{(t)}=0. \quad \text{(C26)}$$

Because of the factor $(n+1-m)$, the sum can trivially be extended up to $n+1$, hence

$$(n+1)\sum_{m=0}^{n+1} c_{n+1,m}\rho_t^{2m}F_{2m+1}^{(t)}=\sum_{m=1}^{n+1} mc_{n+1,m}\rho_t^{2m}F_{2m+1}^{(t)}. \quad \text{(C27)}$$

The lhs vanishes by definition [cf. Eq. (C23)], hence the rhs vanishes and the proof is completed.

The above lemma is instrumental in evaluating the difference between the predictions originated by two subsequent truncations. By applying once more the definition of $F_{2m+1}^{(t)}$, one can easily show that

$$\sum_{m=0}^{n} c_{n,m}[\rho_{t+1}^{2m}(F_{2m+1}^{(t+1)}-F_{2m+1}^{(t)})+(\rho_{t+1}^{2m}-\rho_t^{2m})F_{2m+1}^{(t)}]=0 \quad \text{(C28)}$$

for all $n>t$. Let us now expand the equation to first order in the difference $\rho_{t+1}^2-\rho_t^2$, and make explicit use of the lemma to obtain

$$\sum_{m=t}^{n} c_{n,m}\rho_t^{2m}(F_{2m+1}^{(t+1)}-F_{2m+1}^{(t)})\cong 0 \quad \text{(C29)}$$

for all $n>t$. It is crucial that $F_{2m+1}^{(t+1)}-F_{2m+1}^{(t)}=0$ for all $m<t$.

The equation we obtained allows us to express (within the approximation) all differences $F_{2m+1}^{(t+1)}-F_{2m+1}^{(t)}$ in terms of the single quantity

$$\delta F_t \equiv F_{2t+1}-F_{2t+1}^{(t)}. \quad \text{(C30)}$$

Knowledge of the $c_{n,m}$ and some ingenuity lead to the explicit solution of Eq. (C29):

$$F_{2m+1}^{(t+1)}-F_{2m+1}^{(t)}\cong d_{t,m}\frac{\delta F_t}{\rho_t^{2(m-t)}}, \quad \text{(C31)}$$

where, for all $m>t$,

$$d_{t,m}=\frac{(-1)^{m-t}}{(m-t)!}(2\beta t-\gamma)\prod_{k=1}^{m-t-1}(2\beta m-\gamma-k), \quad \text{(C32)}$$

and obviously $d_{t,t}=1$.

As a corollary to this result, by comparing Eq. (C29) when $t=1$ to Eq. (C24) when $t=2$, we may write down a closed-form expression for all $F_{2m+1}^{(2)}$ coefficients $(m\geqslant 1)$:

$$F_{2m+1}^{(2)}=\frac{1}{6m}\frac{d_{1,m}}{\rho_2^{2(m-1)}}, \quad \text{(C33)}$$

which completes our analysis of the linear parametric model.

One may also show that δF_t is related to the variation of ρ_t by the (linearized) relation

$$\delta F_t\cong\frac{1-2\beta}{2\beta t-\gamma}\left(\sum_{m=0}^{t} m^2 c_{t,m}\rho_t^{2(m-t-1)}F_{2m+1}\right)(\rho_{t+1}^2-\rho_t^2). \quad \text{(C34)}$$

Our numerical estimates, presented in Table XII, show that $\rho_{t+1}^2-\rho_t^2$ is indeed small ($\lesssim 0.01$).

In order to evaluate amplitude ratios, as shown in Appendix B, we must also reconstruct the functions $g(\theta)$ and $g_2(\theta)$, by solving Eqs. (B11) and (B12), respectively. These functions may be expanded in even powers of θ, with coefficients that are functions of ρ satisfying the same differential equations as h_{2n+1}, Eqs. (C10). One may show that, for any given truncation $h^{(t)}(\rho,\theta)$ and arbitrary values of ρ,

$$g^{(t)}(\rho,\theta)=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} c_{nm}\rho^{2m}\frac{F_{2m+1}^{(t)}}{2m+2}\right)\theta^{2n+2} +A(1-\theta^2)^{2\beta+\gamma}, \quad \text{(C35)}$$

where A is an integration constant reflecting the arbitrariness in the zero-field value of the free energy. One may also show that for $n\geqslant t$

$$\sum_{m=0}^{n} c_{nm}\rho^{2m}\frac{F_{2m+1}^{(t)}}{2m+2}\sim\frac{(-1)^n}{(n+1)!}\prod_{k=0}^{n}(2\beta+\gamma-k), \quad \text{(C36)}$$

where the terms on the rhs are the coefficients of the Taylor expansion for $(1-\theta^2)^{2\beta+\gamma}$. As a consequence the constant A may always be chosen such that $g^{(t)}(\rho,\theta)$ is truncated to $O(\theta^{2t})$ for any arbitrary choice of ρ.

In turn, one may also prove that, for any $h^{(t)}(\rho,\theta)$,

$$g_2^{(t)}(\rho,\theta)=\sum_{n=0}^{\infty}\left[\sum_{m=0}^{n} c_{nm}\rho^{2m}(2m+1)F_{2m+1}^{(t)}\right]\theta^{2n}. \quad \text{(C37)}$$

Now, according to Eqs. (C23) and (C24), when we choose for ρ the globally stationary value ρ_t, the coefficients in square brackets vanish for all $n\geqslant t$. As a consequence, for any t the value ρ_t insures the truncation of $g_2^{(t)}(\rho_t,\theta)$ to $O(\theta^{2t-2})$. Thus a unique feature of ρ_t is the simultaneous and consistent truncation of $h(\theta)$ and $g_2(\theta)$.

Notice that we might start by imposing a global stationarity condition directly on a truncated $g_2(\rho,\theta)$, obtaining a different stationary value for ρ, and make use of Eq. (B12) in order to reconstruct the corresponding $h(\theta)$. However, in this case, since $h(\theta)$ must be an odd function of θ, there is no arbitrary integration constant (which is physically a trivial consequence of the definition of a reduced temperature) and therefore $h(\theta)$ cannot be truncated. The resulting parametric model is mathematically consistent, but in practice unappealing, because the calculation of θ_0 from the equation $h(\theta_0)=0$ and the evaluation of universal amplitude ratios becomes quite cumbersome.

The above described formalism can be usefully employed in the context of the ϵ expansion of the critical equation of state. Comparison with ϵ expansion results will also shed further light on the meaning and relevance of the results derived by the prescription of global stationarity.

Our starting point will be the result of Wallace and Zia [27], who showed that, when appropriate conditions are imposed on the zeroth-order approximation, the parametric form of the critical equation of state is automatically truncated in the powers of θ^2 when expanded in the parameter $\epsilon=4-d$. For easier comparison, note that the parameter b introduced by Schofield [24] and used by Wallace and Zia is the same as our θ_0, and the variable change from θ_0 to ρ poses no conceptual problem.

In our reformulation, one may state that, within the ϵ expansion, it is possible to choose to lowest order a value ρ_0 in such a way that, expanding the parametric equation of state in θ and ϵ, one finds, for all $n\geqslant 2$,

$$h_{2n+1}(\rho_0)=O(\epsilon^{n+1}), \tag{C38}$$

and this property should survive the replacement $\rho_0\rightarrow\rho_0+O(\epsilon)$.

As a first application of our formalism, we can verify the consistency of the above statements by checking that, for all $n\geqslant 2$, the condition

$$\sum_{m=0}^{n} c_{n,m}\rho_0^{2m}F_{2m+1}=O(\epsilon^{n+1}) \tag{C39}$$

implies

$$\sum_{m=1}^{n} mc_{n,m}\rho_0^{2m}F_{2m+1}=O(\epsilon^{n}). \tag{C40}$$

The proof is by induction. Assuming the property to hold for a given n, and exploiting the fact that $2\beta-1=O(\epsilon)$, we obtain

$$\sum_{m=0}^{n} [(2\beta-1)m-\gamma+n]c_{n,m}\rho_0^{2m}F_{2m+1}=O(\epsilon^{n+1}). \tag{C41}$$

The initial condition, corresponding to the case $n=1$, has the explicit form

$$\gamma(\gamma-1)+\frac{1}{6}(2\beta-\gamma)\rho_0^2=O(\epsilon^2), \tag{C42}$$

and is a definition of ρ_0. Notice that $\rho_0=\lim_{\epsilon\rightarrow 0}\rho_2$, and in the Ising model $\rho_0^2=2$.

By applying the recursion equations we then obtain

$$\sum_{m=0}^{n} (n+1-m)c_{n+1,m}\rho_0^{2m}F_{2m+1}=O(\epsilon^{n+1}). \tag{C43}$$

The sum can trivially be extended to $n+1$ and, recalling the hypothesis, we obtain

$$\sum_{m=1}^{n+1} mc_{n+1,m}\rho_0^{2m}F_{2m+1}=O(\epsilon^{n+1}), \tag{C44}$$

thus completing the proof. Along the same lines it is straightforward to prove that, for all $n\geqslant 2$,

$$\sum_{m=1}^{n} m^k c_{n,m}\rho_0^{2m}F_{2m+1}=O(\epsilon^{n-k+1}) \tag{C45}$$

for all integers $k\leqslant n$. The initial condition $(n=k)$ is trivially satisfied for all $n\geqslant 2$:

$$\sum_{m=1}^{n} m^n c_{n,m}\rho_0^{2m}F_{2m+1}=O(\epsilon). \tag{C46}$$

As a consequence, the more general statement

$$h_{2n+1}(\rho)=O(\epsilon^{n+1}) \tag{C47}$$

holds for all ρ admitting an ϵ expansion and possessing the limit $\lim_{\epsilon\rightarrow 0}\rho=\rho_0$.

This relation implies, in turn, that, by expanding in ϵ the coefficients F_{2m+1} for $m\geqslant 2$ according to

$$F_{2m+1}=\sum_{k=1}^{\infty} f_{mk}\epsilon^k, \tag{C48}$$

when the f_{mk} for $m<k$ are known, then all f_{mk} for $m\geqslant k$ are fully determined.

As a simple application of the above, we obtained the following closed-form result:

$$f_{m1}=\frac{(-1)^m}{m(m-1)}\rho_0^{-2m}\lim_{\epsilon\rightarrow 0}\frac{\gamma-1}{\epsilon}, \tag{C49}$$

where $\gamma\cong 1+\frac{1}{2}\epsilon$ and $\rho_0=\sqrt{2}$.

Let us now consider the linear parametric model with global stationarity in the context of the ϵ expansion: ρ_2 satisfies the condition $\rho_2=\rho_0+O(\epsilon)$, though it does not coincide (and is not expected to) with the ϵ-expanded ρ value adopted by Guida and Zinn-Justin [23].

Now note that for any higher-order truncation the stationarity condition is still solved by $\rho_t=\rho_0+O(\epsilon)$, as shown explicitly by the above-derived Eq. (C41). As a consequence, any stationary truncation is an accurate description of the ϵ-expanded parametric equation of state up to $O(\epsilon^t)$ included. Actually, the freedom to choose ρ leaves such an expansion highly underdetermined, and many other prescriptions might work, including that of fixing ρ (or, alternatively, θ_0) to its zeroth-order value. It is, however, certainly pleasant to recognize that our approach based on stationarity falls naturally into the set of consistent truncations. As a side remark, note that all the coefficients of the ϵ expansion of ρ_t will, in general, be changed order by order in t, and will also, in general, be complex numbers. This fact will by no means affect the real character of the expanded physical amplitudes, and will not even prevent ρ and θ_0 from taking real values in the actual three-dimensional calculations.

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